Faculty Of Graduate Studies

Mathematics Program

Qualitative Analysis of Solutions of Some Systems of Difference Equations.

Prepared By :

Mohammad Bader.

Supervised By:

Prof. Marwan Aloqeili.

M.Sc.Thesis

Birzeit University

Palestine

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This thesis was submitted in partial fulfillment of the requirements for the Master's Degree in Mathematics from the Faculty of Graduate Studies at Birzeit University, Palestine.

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This thesis was defended on................. And approved by:

Committee Members :

1.Prof. Marwan Aloqeili (Head of committee) ....

2. Prof. Mohammad Saleh (Internal Examiner)...................

3. Dr. Hani Kabajah (Internal Examiner) ....

الإهداء

بسم الله الرحمن الرحيم :

(ولولا فضل الله عليك ورحمته لهمت طائفة منهم أن يضلوك وما يضلون إلا أنفسهم وما يضرونك من شيء وأنزل الله عليك الكتاب والحكمة وعلمك ما لم تكن تعلم وكان فضل الله عليك عظيما). صدق الله العظيم [النساء:١١٣].

لا يسعني و أنا أخط اخر اللمسات في هذه الدراسة إلا أن أتقدم بالشكر إلى كل من كانت له فيها مساهمة و لو بسيطة.

وأخص بالشكر البرفسور مروان العقيلي و الذي بدوره كان له الفضل بعد الله عزوجل في إنارة طريق البحث لي من خلال تعليماته و توجيهاته.

كما أتقدم بالشكر إلى لجنة النقاش البرفسور محمد صالح و الدكتور هاني كباجه الذين بدورهم سهلوا لي مهمة إنجاز هذه الرسالة.

أهدي ثمرة جهدي المتواضع إلى من وهبوني الحياة و الأمل و النشأة على شغف الإطلاع و المعرفة ومن علموني أن أرتقي سلم الحياة بحكمة و صبر : والدي العزيز والدتي العزيزة.

وأخيرا إلى كل طالب علم سعى بعلمه ليفيد الإسلام و المسلمين بكل ما أعطاه الله من علم و معرفة.

DECLARATION

I certify that this Thesis, submitted for the degree of Master of Mathematics to the Department of Mathematics at Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis (or any part of it) has not been submitted for a higher degree to any other university or institution.

Mohammad Bader	Signature
June, 2021	....

ABSTRACT

In this thesis, we investigate semi-cycles, boundedness, persistence of positive solutions, and global asymptotic stability of the unique positive equilibrium of two different systems of two nonlinear difference equations.
The first system is:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \cdots$$

with parameters A, B are positive real numbers, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.
The second system is:

$$x_{n+1} = A + \frac{x_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-k}}, \quad n = 0, 1, \cdots$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.

الملخص

في هذه الرسالة، نقوم بدراسة الخصائص الديناميكية للحلول الموجبة لنظامين من معادلتي فرق من الدرجات العليا. النظام الأول هو:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \cdots$$

حيث أن A, B هما عددان حقيقيان موجبان و $x_i, y_i \in (0, \infty)$ لكل $i = -k, -k+1, \cdots, 0$ حيث k عدد صحيح موجب . بينما النظام الثاني هو:

$$x_{n+1} = A + \frac{x_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-k}}, \quad n = 0, 1, \cdots$$

حيث أن $A, B \in (0, \infty)$ و x_i, y_i هي أعداد حقيقية موجبة لكل $i = -k, -k+1, \cdots, 0$ و $k \in Z^+$.

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1. PRELIMINARIES

1.1 Introduction

Difference equations and discrete dynamical systems have received attention from researchers in particular mathematical model which studies problems in physics, economics, engineering and biology. These equations and systems can help to develop the theory of difference equations. Difference equations which may It difficult to completely recognize the behavior of their solutions.

Recently, nonlinear difference equations and systems are of extensive interest[[3] [8],[11],[27]].

Particularly, in 1998, Papaschinopoulos and Schinas [18] proved that any positive solution of the following system of difference equations oscillates about the equilibrium:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \ldots \tag{1.1.1}$$

where $A > 0$ and p, q are positive integers. They proved that any positive solution of (1.1.1) oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$, and if $A > 0$ and at least one of p, q is an odd number (respectively, $A > 1$ and p, q are both even numbers), then any positive solution of (1.1.1) is bounded. Moreover, they showed that when $A > 1$ therefore the positive unique equilibrium of the system (1.1.1) is globally asymptotic stable. Moreover, they considered system in the case that $A = 0$ and $p = q = 1$, and found that every solution of system (1.1.1) in this case is periodic of period 6 .

After that, in 2000 , Papaschinopoulos and Schinas [19] investigated the system:

$$x_{n+1} = A + \frac{x_{n-1}}{y_n}, \quad y_{n+1} = A + \frac{y_{n-1}}{x_n}, \quad n = 0, 1, \ldots \tag{1.1.2}$$

where A is a positive constant and x_{-1}, x_0, y_{-1}, y_0 are positive numbers.They proved that any positive solution of the system oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$.

Moreover, that system (1.1.2)has been proved as having a periodic solution of period two if $A = 1$, and that any positive solution of system (1.1.2) tends to the equilibrium as $n \to \infty$.

Furthermore, they showed that if $0 < A < 1$, then system (1.1.2) has unbounded solutions. If $A = 1$, then every positive solution of (1.1.2) tends to a periodic solution of period two, and if $A > 1$ then the positive equilibrium $(\bar{x}, \bar{y}) = (A+1, A+1)$ of (1.1.2) is globally asymptotically stable.

Whereas Papaschinopoulos and Papadopoulos [17] studied, in 2002, the existence of positive solutions of the equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-m}}, \quad n = 0, 1, \ldots \tag{1.1.3}$$

And they found both bounded and unbounded solutions of (1.1.3). They also investigated The difference equations of the following system:

$$x_{n+1} = A + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \ldots \tag{1.1.4}$$

where $m \in \{1, 2, \ldots\}$, and $x_{-m}, x_{-m+1}, \ldots, x_0, y_{-m}, y_{-m+1}, \ldots, y_0$ are positive constants and A, B are positive real numbers. They proved that in case that $A > 1$ and $B > 1$, the solution of (1.1.4) is bounded and persists, and there will be a unique positive equilibrium $(\bar{x}, \bar{y})$ of system (1.1.4) and that every positive solution

of (1.1.4) tends to that unique positive equilibrium as $n \to \infty$. They could also found unbounded solutions when $0 < A < 1$ or $0 < B < 1$.
In 2004, Camouzis and Papaschinopoulos [4] had studied the persistence and boundedness of mentioned positive solutions of the following systems:

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \dots \tag{1.1.5}$$

where x_i, y_i are positive numbers for $i = -m, -m+1, \dots, 0$ and m is a positive integer. Furthermore, they proved that (1.1.5) has an infinite number of positive equilibrium solutions and that every positive solution converges to a positive equilibrium solution $(\bar{x}, \bar{y}) = (2, 2)$ as $n \to \infty$.

In $2007, Y$. Zhang et al. [27] investigated the system:

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad n = 0, 1, \dots \tag{1.1.6}$$

with positive parameter A, the initial conditions x_i, y_i are positive real numbers for $i = -m, -m+1, \dots, 0$, and m is a positive integer. Zhang et al. proved that the unique positive equilibrium of (1.1.6) is globally asymptotically stable for $A > 1$, and the positive solution of system (1.1.6) is bounded and persists when $A \geq 1$, they also found unbounded solutions of system (1.1.6) when $0 < A < 1$, and showed that for $A = 1$, if m is odd then any positive solution of (1.1.6) with prime period two is of the form

$$\cdots, (b, b), \left(\frac{b}{b-1}, \frac{b}{b-1}\right), (b, b), \left(\frac{b}{b-1}, \frac{b}{b-1}\right), \dots$$

where $1 < b \neq 2$, however, if m is even then any positive solution of (1.1.6) with prime period two takes the form

$$\cdots, \left(b, \frac{b}{b-1}\right), \left(\frac{b}{b-1}, b\right), \left(b, \frac{b}{b-1}\right), \left(\frac{b}{b-1}, b\right), \dots$$

where $1 < b \neq 2$.

While in 2013, the concept of the global asymptotic stability of positive equilibrium and persistence and boundedness of positive solutions were studied by Q. Zhang, Yang, and Liu [26] of the following system:

$$x_{n+1} = A + \frac{x_{n-m}}{y_n}, \quad y_{n+1} = B + \frac{y_{n-m}}{x_n}, \quad n = 0, 1, \dots \tag{1.1.7}$$

where $A, B, x_i, y_i \in (0, \infty)$ for $i = -m, -m+1, \dots, 0$ and $m \in \mathbb{Z}^+$. They found unbounded solutions for system (1.1.7) when A and B are less than one, and proved that when $A \geq 1$ and $B \geq 1$ the positive solution of system (1.1.7) is bounded and persists, and when $A > 1$ and $B > 1$ the positive equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{AB-1}{B-1}, \frac{AB-1}{A-1}\right)$ is globally asymptotically stable.
A year later, the concept of global asymptotic behavior of the system including two rational difference equation were demonstrated and studied by Q. Zhang et al [25]:

$$x_{n+1} = A + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}, \quad y_{n+1} = B + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}, \quad n = 0, 1, \dots \tag{1.1.8}$$

where A, B, x_i, y_i are positive real numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$. More precisely, Zhang et al. proved that if $A > \frac{1}{k}$ and $B > \frac{1}{k}$, therefore each positive solution of system (1.1.8) is bounded and persists. Moreover, they proved that every positive solution converges to the positive equilibrium $(\bar{x}, \bar{y})$ as $n \to \infty$.
Finally, D. Zhang et al. [24] presented and studied the system

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \dots \tag{1.1.9}$$

with considering parameters $A > 0$, the initial conditions x_i, y_i are arbitrary positive real numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$. The above mentioned scientists investigated the asymptotic behavior of positive solutions of the system in the cases $0 < A < 1, A = 1$ and $A > 1$. When $0 < A < 1$, they might discover unbounded solutions of system (1.1.9), and they proved when $A = 1$ the system (1.1.9) can have two periodic solutions, and every positive solution is bounded and persists. They additionally show that the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A+1, A+1)$ is a global attractor when $A > 1$.

The semi cycle of the positive solutions of the system were being investigated by Gumus [12] in 2018, and when $A > 1$ show that the unique positive equilibrium point $(\bar{x}, \bar{y}) = (A + 1, A + 1)$ is globally asymptotically stable.
In 2019, S. Abualrob and M. Aloqeili,[[2],[1]] investigated semi cycle , boundedness and the persistence of solutions that are positive and the unique positive equilibrium of the two different systems of the two nonlinear difference equation that are related to global asymptotic stability.
the first system is:

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n} \quad n = 0, 1, \dots \tag{1.1.10}$$

with considering parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k + 1, \dots, 0$ and $k \in \mathbb{Z}^+$.
The second system is:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}} \quad n = 0, 1, \dots \tag{1.1.11}$$

with parameters $A > 0$ and the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k + 1, \dots, 0$ and $k \in \mathbb{Z}^+$.

Other associated difference equations and systems may be located in references [[3],[5]-[8],[11],[13]-[16],[20]-[23]].
More details when considering the theory of difference equations have been supplied in [[9],[10]]. Motivated by all the systems we previously mentioned, we introduce in Chapter 2 the system

$$x_{n+1} = A + \frac{y_n}{x_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \dots$$

with positive parameters A and B, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k + 1, \dots, 0$ and $k \in \mathbb{Z}^+$.
In Chapter 3 , we introduce the system

$$x_{n+1} = A + \frac{x_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-k}}, \quad n = 0, 1, \dots$$

with positive parameters A and B, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \dots, 0$ and $k \in \mathbb{Z}^+$. As a long way as we know, no work has been reported in the literature on the dynamics of those system.
In Chapter 2, the semi-cycle of the system of the positive solutions of system (2.0.1) is studied, when $0 < A < 1$ and $0 < B < 1$ we also find unbounded solutions of the same system. When $A \geq 1$ and $B \geq 1$ we prove that the positive solutions of system (2.0.1) are bounded and persist. Finally, we show that if $A > 1$ and $B > 1$ then the unique positive equilibrium of system (2.0.1) is globally asymptotically stable.

Moreover, in Chapter 3 , we investigate system (3.0.1) via the method of semi-cycle analysis , and then we assume some conditions to get unbounded solutions for this system. We also prove that if $A \geq 1$ and $B \geq 1$ then every positive solutions of system (3.0.1) are bounded and persist. Then, we show that when $A > 1$ and $B > 1$ the positive equilibrium point of system (3.0.1) is globally asymptotically stable.

We conclude each of these two chapters by numerical examples that illustrate the our results.

1.2 Basic Definitions and Results

In this part, we provide basic definitions and results that we're about to use in the following chapters. Consider the $2(k+1)$ -dimensional dynamical system of the following form:

$$\begin{aligned} x_{n+1} &= f\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right) \\ y_{n+1} &= g\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, \ldots \end{aligned} \tag{1.2.1}$$

where f, g are continuously differentiable real valued functions. For example

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots$$

Definition 1.1. *(Equilibrium Point). A point $(\bar{x}, \bar{y})$ is said to be an equilibrium point of system (1.2.1) if*

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y}) \\ \text{and} \quad \bar{y} &= g(\bar{x}, \bar{x}, \ldots, \bar{x}, \bar{y}, \bar{y}, \ldots, \bar{y}) \end{aligned} \tag{1.2.2}$$

Example 1.1. Consider the system

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots$$

To find equilibrium point we solve $f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ implies $\bar{x} = A + \frac{\bar{y}}{\bar{y}} = A + 1$ and $\bar{y} = B + \frac{\bar{x}}{\bar{x}} = B + 1$, so $(\bar{x}, \bar{y}) = (A+1, B+1)$.

Definition 1.2. *(Stable, Unstable, Attracting, Asymptotically Stable and Globally Asymptotically Stable Equilibrium Point). If $(\bar{x}, \bar{y})$ is an equilibrium point of (1.2.1) then*

1. $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition $(x_i, y_i), i \in \{-k, -k+1, \ldots, 0\}$ if $\left\|\sum_{i=-k}^{0} (x_i, y_i) - (\bar{x}, \bar{y})\right\| < \delta$ implies that for all $n > 0$, $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$, where $\|.\|$ is usual Euclidian norm in $\mathbb{R}^2$. Otherwise, $(\bar{x}, \bar{y})$ is called unstable.

2. An equilibrium point $(\bar{x}, \bar{y})$ is called attracting if there exists $\eta > 0$ such that

$$\left\|\sum_{i=-k}^{0} (x_i, y_i) - (\bar{x}, \bar{y})\right\| < \eta \text{ implies } \lim_{n\to\infty} (x_n, y_n) = (\bar{x}, \bar{y}) \tag{1.2.3}$$

3. $(\bar{x}, \bar{y})$ is called a global attractor if in $2, \eta = \infty$.

4. An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotically stable if it is both stable and attracting, and it is said to be globally asymptotically stable if it is both stable and global attractor.

Definition 1.3. *(Positive Solution). A pair of sequences of positive real numbers $\{x_n, y_n\}_{n=-k}^{\infty}$ that satisfies (1.2.1) is a positive solution of (1.2.1).*

Definition 1.4. *(Equilibrium Solution). If a positive solution of (1.2.1) is a pair of constants $(\bar{x}, \bar{y})$, then the solution is the equilibrium solution.*

Definition 1.5. *(Periodic Solution). A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.2.1) is said to be periodic if there exists a positive integer m, such that for all $n \geq -k, (x_n, y_n) = (x_{n+m}, y_{n+m})$. m is called the period of the solution.*

Definition 1.6. *(Eventually Periodic Solution). A positive solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of (1.2.1) is said to be eventually periodic if there exist an integer $l > -k$ and a positive integer m, such that $(x_{n+l}, y_{n+l}) = (x_{n+l+m}, y_{n+l+m})$ for all $n = 0, 1, \ldots$ where m is the period of the solution.*

Definition 1.7. *(Bounded Solution). A positive solution* $\{x_n, y_n\}_{n=-k}^{\infty}$ *of (1.2.1) is bounded and persists if there exist positive real numbers* P_1, Q_1, P_2 *and* Q_2 *such that* $P_1 \leq x_n \leq Q_1$ *and* $P_2 \leq y_n \leq Q_2$ *for* $n \geq -k$.

Definition 1.8. *(Increasing and Decreasing Solution). A positive solution* $\{x_n, y_n\}_{n=-k}^{\infty}$ *of (1.2.1) is demonstrated to be increasing (respectively decreasing) if* $n > m$, *then* $x_n > x_m$ *and* $y_n > y_m$ *(respectively* $x_n < x_m$ *and* $y_n < y_m$*) for all* $n \geq 1$ *and* $m \geq 1$.

Definition 1.9. *A series of consecutive expression* $\{x_t, \dots, x_r\}$ *(respectively* $\{y_t, \dots, y_r\}$*)*, $t \geq -k$, *and* $r \leq \infty$ *is demonstrated to be a positive semi-cycle if* $x_i \geq \bar{x}$ *(respectively* $y_i \geq \bar{y}$*)*, $i \in \{t, \dots, r\}$, $x_{t-1} < \bar{x}$ *(respectively* $y_{t-1} < \bar{y}$*), and* $x_{r+1} < \bar{x}\,(y_{r+1} < \bar{y})$

Definition 1.10. *A series of consecutive expression* $\{x_t, \dots, x_r\}$ *(respectively* $\{y_t, \dots, y_r\}$*)*, $t \geq -k$, *and* $r \leq \infty$ *is demonstrated to be a negative semi-cycle if* $x_i < \bar{x}$ *(respectively* $y_i < \bar{y}$*)*, $i \in \{t, \dots, r\}, x_{t-1} \geq \bar{x}$ *(respectively* $y_{t-1} \geq \bar{y}$*), and* $x_{r+1} \geq \bar{x}$ $(y_{r+1} \geq \bar{y})$

Definition 1.11. *A series of consecutive expression* $\{(x_t, y_t), \dots, (x_r, y_r)\}$, $t \geq -k$, *and* $r \leq \infty$ *is demonstrated to be a positive semi-cycle (respectively negative semi-cycle) if both* $\{x_t, \dots, x_r\}$ *and* $\{y_t, \dots, y_r\}$ *are positive semi-cycles (respectively negative semi-cycles).*

Definition 1.12. *A series of consecutive expression* $\{(x_t, y_t), \dots, (x_r, y_r)\}$, *and* $t \geq -k$, $r \leq \infty$ *is demonstrated to be a positive semi-cycle (respectively negative semi-cycle) with related to* x_n *and negative semi-cycle(respectively positive semi-cycle) with related to* y_n *if* $\{x_t, \dots, x_r\}$ *is a positive semi-cycle (respectively negative semi-cycle) and* $\{y_t, \dots, y_r\}$ *is a negative semi-cycle (respectively positive semi-cycle).*

The first semi-cycle of a solution of (1.2.1) starts with the term (x_{-k}, y_{-k}), *and it's positive (respectively negative) if* $x_{-k} \geq \bar{x}$ *and* $y_{-k} \geq \bar{y}$ *(respectively* $x_{-k} < \bar{x}$ *and* $y_{-k} < \bar{y}$*)*

Definition 1.13. *(Nonoscillatory Solution). A sequence x_n (respectively y_n) is called nonoscillatory about $\bar{x}$ (respectively $\bar{y}$) if there exists $N \geq -k$ such that $x_n \geq \bar{x}$ (respectively, . $y_n \geq \bar{y}$) or $x_n < \bar{x}$ (respectively, $y_n < \bar{y}$) for all $n \geq N$.*
We mention that a solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of system (1.2.1) is a nonoscillatory solution about $(\bar{x}, \bar{y})$ if x_n is nonoscillatory about $\bar{x}$ and y_n is nonoscillatory about $\bar{y}$. However, a solution $\{x_n, y_n\}_{n=-k}^{\infty}$ is called oscillatory if it is not nonoscillatory.

Definition 1.14. *(Nonoscillatory Positive and Nonoscillatory negative Solutions). A solution $\{x_n, y_n\}_{n=-k}^{\infty}$ of system (1.2.1) is a nonoscillatory positive (respectively negative) solution about $(\bar{x}, \bar{y})$ if there exists $N \geq -k$ such that $x_n \geq \bar{x}$ and $y_n \geq \bar{y}$ (respectively $x_n < \bar{x}$ and $y_n < \bar{y}$) for all $n \geq N$.*

Definition 1.15. *(Jacobian Matrix). The Jacobian Matrix is a matrix that takes the partial derivatives of the linearization with respect to each of the sequence at the equilibrium point.*

Definition 1.16. *(Linearized Form of (1.2.1) Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1.2.1) where f, g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (1.2.1) concerning the point of equilibrium has the form:*

$$X_{n+1} = JX_n$$

where $X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k})^T$ and J *is a Jacobian matrix of system (1.10) concerning the point of equilibrium.*

Theorem 1.1. *[28] For the linearized system $X_{n+1} = JX_n, n = 0, 1, \ldots$ of (1.2.1). If all eigenvalues of the Jacobian matrix J about $(\bar{x}, \bar{y})$ lie inside the open unit disk $|\lambda| < 1$, then $(\bar{x}, \bar{y})$ is locally asymptotically stable. If one of them has a modulus greater than one, then $(\bar{x}, \bar{y})$ is unstable.*

Definition 1.17. *(Limit Superior and Limit Inferior). Let* $\{x_n\}$ *be a sequence of real numbers. The limit superior of* $\{x_n\}$ *, denoted by* $\limsup \{x_n\}$ *, is defined by*

$$\limsup \{x_n\} = \lim_{n\to\infty} [\sup \{x_m; m \geq n\}] = \inf_{n\geq 0} [\sup \{x_m; m \geq n\}]$$

The limit inferior of $\{x_n\}$ *, denoted by* $\liminf \{x_n\}$ *, is defined by*

$$\liminf \{x_n\} = \lim_{n\to\infty} [\inf \{x_m; m \geq n\}] = \sup_{n\geq 0} [\inf \{x_m; m \geq n\}]$$

Example 1.2. Consider the sequence $\{x_n\} = \{0, 1, 0, 1, ...\}$. Then $\beta_n = \sup\{x_m, m \geq n\} = 1$ and $\alpha_n = \inf\{x_m, m \geq n\} = 0$

Example 1.3. Consider the sequence $\{y_n\} = (-1)^n$. Then $\beta_n = \sup\{y_m, m \geq n\} = 1$ and $\alpha_n = \inf\{y_m, m \geq n\} = -1$

Definition 1.18. *(Spectral Radius). Let* M *be any real* $n \times n$ *matrix, and assume* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* M*. Then the spectral radius of* M*, denoted by* $\rho(M)$*, is given by:*

$$\rho(M) = \max_{1\leq i\leq n} \{|\lambda_i|\}$$

Theorem 1.2. *[28] Let* $||.||$ *be any matrix norm defined on the set of all real* $n \times n$ *matrices* $(\mathcal{M}_n)$ *. Then for any matrix* $M \in \mathcal{M}_n$

$$\rho(M) \leq \|A\|$$

Definition 1.19. *(Infinite Norm of a Matrix). Let* M *ba any matrix in* $\mathcal{M}_n$*. The infinite norm of* M*, denoted by* $\|M\|_\infty$*, is given by:*

$$\|M\|_\infty = \max_{1\leq r\leq n} \sum_{c=1}^{n} |m_{r,c}|$$

2. DYNAMICS OF THE SYSTEM $X_{N+1} = A + \frac{Y_N}{Y_{N-K}}, \quad Y_{N+1} = B + \frac{X_N}{X_{N-K}}$

In this chapter, we introduce the symmetrical system:

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \cdots \tag{2.0.1}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$. We observe the dynamical behavior of this system in the cases: When $0 < A < 1$ and $0 < B < 1$ and when $A > 1$ and $B > 1$, we additionally look at the behavior of the positive solutions of (2.0.1) using the semi-cycle analysis method. Finally, we give some numerical examples that supports the results in this chapter.

System (2.0.1) has the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ since $f(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ implies $\bar{x} = A + \frac{\bar{y}}{\bar{y}} = A + 1$ and $\bar{y} = B + \frac{\bar{x}}{\bar{x}} = B + 1$ so $(\bar{x}, \bar{y}) = (A+1, B+1)$.

There are two cases to be considered:

- Case 1: If $A = B$ then system (2.0.1) turns into the symmetrical system

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n = 0, 1, \cdots$$

  with parameter $A > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$, which was studied in [1].

- Case 2: We study the general case, which is a generalization of the study in [1].

2.1 Semi-cycle Analysis I

In this section, we have a look at the behavior of positive solutions of system (2.0.1) by semi-cycle analysis method.

Theorem 2.1. *Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution of system (2.0.1). Then, both this solution is non-oscillatory solution or it oscillates about the equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ with semi-cycles such that if there exists a semi-cycle with at least k terms, then each semi-cycle after that has at least $k+1$ terms.*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is a solution of system (2.0.1), and there exists an integer $n_0 \geq 0$ such that (x_{n_0}, y_{n_0}) is the last term of a semi-cycle that has at least k terms. Then, both

$$..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} < 1 + A \leq x_{n_0+1}$$

and

$$..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} < 1 + B \leq y_{n_0+1}$$

or

$$..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} \geq 1 + A > x_{n_0+1}$$

and

$$..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} \geq 1 + B > y_{n_0+1}$$

- Case 1: If $..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} < 1+A \leq x_{n_0+1}$ and $..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} < 1 + B \leq y_{n_0+1}$, then

$$x_{n_0+2} = A + \frac{y_{n_0+1}}{y_{n_0-k+1}} > A + 1 \text{ and } y_{n_0+2} = B + \frac{x_{n_0+1}}{x_{n_0-k+1}} > B + 1$$

$$x_{n_0+3} = A + \frac{y_{n_0+2}}{y_{n_0-k+2}} > A + 1 \text{ and } y_{n_0+3} = B + \frac{x_{n_0+2}}{x_{n_0-k+2}} > B + 1$$

$$\vdots$$

$$x_{n_0+k} = A + \frac{y_{n_0+k-1}}{y_{n_0-1}} > A + 1 \text{ and } y_{n_0+k} = B + \frac{x_{n_0+k-1}}{x_{n_0-1}} > B + 1$$

$$x_{n_0+k+1} = A + \frac{y_{n_0+k}}{y_{n_0}} > A + 1 \text{ and } y_{n_0+k+1} = B + \frac{x_{n_0+k}}{x_{n_0}} > B + 1$$

hence, the semi-cycle beginning with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume the semi-cycle which begins with (x_{n_0+1}, y_{n_0+1}) has exactly $k+1$ terms, then the following semi-cycle will begin with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that
$x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+k+1} \geq 1 + A > x_{n_0+k+2}$ and $y_{n_0+1}, y_{n_0+2}, ..., y_{n_0+k+1} \geq 1 + B > y_{n_0+k+2}$, then for $i = 1, 2, 3, .., k$

$$x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} < A + 1$$

and

$$y_{n_0+k+2+i} = B + \frac{x_{n_0+k+1+i}}{x_{n_0+1+i}} < B + 1$$

so, each semi-cycle after this point must have at least $k+1$ terms.

- Case 2: If $..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} \geq 1+A > x_{n_0+1}$ and $..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} \geq 1 + B > y_{n_0+1}$, then for all $i = 2, 3, ..., k+1$

$$x_{n_0+i} = A + \frac{y_{n_0-1+i}}{y_{n_0-k-1+i}} < A + 1$$

and

$$y_{n_0+i} = B + \frac{x_{n_0-1+i}}{x_{n_0-k-1+i}} < B + 1$$

hence, the semi-cycle beginning with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume this semi-cycle has exactly $k+1$ terms, then the following semi-cycle will begins with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that $x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+k+1} < 1 + A \leq x_{n_0+k+2}$ and $y_{n_0+1}, y_{n_0+2}, ..., y_{n_0+k+1} < 1 + B \leq y_{n_0+k+2}$ then for $i = 1, 3, ..., k$

$$x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} > A + 1$$

and

$$y_{n_0+k+2+i} = B + \frac{x_{n_0+k+1+i}}{x_{n_0+1+i}} > B + 1$$

so, each semi-cycle after this point must have at least $k+1$ terms.

$\blacksquare$

Theorem 2.2. *System (2.0.1) has no nontrivial k-periodic solutions of period k (not necessarily prime period k).*

Proof. Assume system (2.0.1) has a k-periodic solution. Then $(x_{n-k}, y_{n-k}) = (x_n, y_n)$ for all $n \geq 0$, and so $x_{n+1} = A + \frac{y_n}{y_{n-k}} = A + 1$ and $y_{n+1} = B + \frac{x_n}{x_{n-k}} = B + 1$, for all $n \geq 0$. Thus, the solution $(x_n, y_n) = (A + 1, B + 1)$ is the equilibrium solution of (2.0.1) ∎

Theorem 2.3. *All non-oscillatory solutions of System (2.0.1)have a tendency to the equilibrium.$(\bar{x}, \bar{y}) = (A + 1, B + 1)$.*

Proof. Assume that system (2.0.1) has a non-oscillatory solution say $\{x_n, y_n\}_{n=-k}^{\infty}$. Then via way of means of Theorem (2.1) the solution includes a single semi-cycle, either this semi-cycle is positive or negative. Assume that the solution is of negative semi-cycle. Then for all $n \geq -k$, $(x_n, y_n) < (A + 1, B + 1)$, so

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} < A + 1 \text{ implies } y_n < y_{n-k}$$

$$y_{n+1} = B + \frac{x_n}{x_{n-k}} < B + 1 \text{ implies } x_n < x_{n-k}$$

$$A < ... < x_{n+k} < x_n < x_{n-k} < A + 1$$

and

$$B < ... < y_{n+k} < y_n < y_{n-k} < B + 1$$

which means that x_n, y_n have k subsequences

$$\{x_{nk}\}, \{x_{nk+1}\}, \cdots, \{x_{nk+(k-1)}\} \textit{ and } \{y_{nk}\}, \{y_{nk+1}\}, \cdots, \{y_{nk+(k-1)}\}$$

every subsequence is decreasing and bounded from below, so every one of them is convergent, so for all $i = 0, 1, ..., k - 1$ there exist α_i, β_i such that

$$\lim_{n\to\infty} x_{nk+i} = \alpha_i \textit{ and } \lim_{n\to\infty} y_{nk+i} = \beta_i.$$

Thus

$$(\alpha_0, \beta_0), (\alpha_1, \beta_1), \cdots, (\alpha_{k-1}, \beta_{k-1})$$

is a k-periodic solution of system (2.0.1), which contradicts Theorem (2.2) except the solution is the trivial solution. So, the solution converges to the equilibrium. ■

Theorem 2.4. *Any increasing solution to system (2.0.1) is non-oscillatory positive (the infinite semi-cycle in the solution is a positive semi-cycle).*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is an increasing non-oscillatory solution to system (2.0.1).Then, either $A+1 \leq x_1$ *and* $B+1 \leq y_1$ *or* $x_1 < A+1$ *and* $y_1 < B+1$.

- Case 1: If $A+1 \leq x_1$ and $B+1 \leq y_1$, since the solution is increasing then $A+1 \leq x_1 \leq x_2 \leq x_3 \leq ...$ and $B+1 \leq y_1 \leq y_2 \leq y_3 \leq ...$, so the solution has an infinite positive semi-cycle.
- Case 2: If $x_1 < A+1$ and $y_1 < B+1$, then we claim that the semi-cycle containing (x_1, y_1) ends with (x_i, y_i) such that $1 \leq i \leq k+1$. If $i = k+2$ then

  $$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A+1 \text{ and } y_{k+2} = B + \frac{x_{k+1}}{x_1} < B+1$$

  imply that

  $$y_{k+1} < y_1 \text{ and } x_{k+1} < x_1$$

  but $k+1 > 1$ which contradicts the fact that the solution is increasing, so any increasing solution of system is non-oscillatory positive.

■

Theorem 2.5. *System (2.0.1)has no decreasing non-oscillatory solution.*

Proof. Assume system (2.0.1) has a decreasing non-oscillatory solution say $\{x_n, y_n\}_{n=-k}^{\infty}$. As in proof of Theorem (2.4) the solution is either of the form

$$... \leq x_3 \leq x_2 \leq x_1 < A + 1 \text{ and } ... \leq y_3 \leq y_2 \leq y_1 < B + 1$$

or there exists a positive integer $n_0 \geq k + 1$, such that

$$... \leq x_{n_0+2} \leq x_{n_0+1} < A + 1 \leq x_{n_0} \leq x_{n_0-1} ...$$

and

$$... \leq y_{n_0+2} \leq y_{n_0+1} < B + 1 \leq y_{n_0} \leq y_{n_0-1} ...$$

where the positive semi-cycle ending with (x_{n_0}, y_{n_0}) will have at most $2k + 2$ terms. In each cases, the solution has an infinite negative semi-cycle which contradicts Theorem (2.3). Hence, system (2.0.1) has no decreasing non-oscillatory solutions. ∎

2.2 Semi-cycle Analysis II

In this section, we observe extra properties of qualitative behavior of positive solutions of system (2.0.1) by semi-cycle analysis.Throughout this section, we carry out semi-cycle analysis when x and y have semi-cycles of the specific types, that is, positive (resp. negative) semi-cycle for x and negative (resp. positive) semi-cycle for y, see definition(1.12) . We call the solution in this situation a solution with specific semi-cycles.

Theorem 2.6. *The following statements are true:*

(a) Any solution to system (2.0.1) that is increasing with respect to x and decreasing with respect to y is non-oscillatory positive with respect to x and non-oscillatory negative with respect to y.

(b) Any solution to system (2.0.1) that is decreasing with respect to x and increasing with respect to y is non-oscillatory a negative semi-cycle with respect to x and non-oscillatory positive with respect to y.

Proof. We prove statement (a). Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is an increasing solution with respect to x and decreasing with respect to y to system (2.0.1). Then we have the following cases:

(1) $A + 1 < x_1$ and $B + 1 \geq y_1$.

(2) $A + 1 < x_1$ and $B + 1 < y_1$.

(3) $A + 1 \geq x_1$ and $B + 1 \geq y_1$.

(4) $A + 1 \geq x_1$ and $B + 1 < y_1$.

- Case (1): if $A + 1 < x_1$ and $B + 1 \geq y_1$, since the solution is increasing with respect to x and decreasing with respect to y then $A + 1 < x_1 < x_2 < x_3 < \ldots$ and $B + 1 \geq y_1 > y_2 > y_3 > \ldots$, so the solution has an infinite positive semi-cycle with respect to x and an infinite negative semi-cycle with respect to y.

- Case (2): if $A+1 < x_1$ and $B+1 < y_1$, then we can conclude that the solution has an infinite positive semi-cycle with respect to x. As for y, we claim that the semi-cycle containing y_1 ends with y_i such that $1 \leq i \leq k+1$. If $i = k+2$, then
$$y_{k+2} = B + \frac{x_{k+1}}{x_1} > B + 1$$
imply that the solution of (2.0.1) has an infinite positive semi-cycle with respect to x and infinite positive semi-cycle with respect to y.

- Case (3): if $A+1 \geq x_1$ and $B+1 \geq y_1$, then we can conclude that the solution has an infinite negative semi-cycle with respect to y. As for x, we claim that the semi-cycle containing x_1 ends with x_i such that $1 \leq i \leq k+1$. If $i = k+2$, then

$$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A + 1$$

imply that the solution of (2.0.1) has an infinite negative semi-cycle with respect to x and infinite negative semi-cycle with respect to y.

- Case (4): if $A + 1 \geq x_1$ and $B + 1 < y_1$, then we claim that the semi-cycle containing (x_1, y_1) ends with (x_i, y_i) such that $1 \leq i \leq k+1$. If $i = k+2$, then

$$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A + 1 \text{ and } y_{k+2} = B + \frac{x_{k+1}}{x_1} > B + 1$$

imply that the solution of (2.0.1) has an infinite negative semi-cycle with respect to x and infinite positive semi-cycle with respect to y.

Now we need prove statement (b). Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is an increasing solution with respect to y and decreasing with respect to x to system (2.0.1). Then we have the following cases:

(1) $A + 1 \geq x_1$ and $B + 1 < y_1$.

(2) $A + 1 < x_1$ and $B + 1 < y_1$.

(3) $A + 1 \geq x_1$ and $B + 1 \geq y_1$.

(4) $A + 1 < x_1$ and $B + 1 \geq y_1$.

- Case (1):if $A + 1 \geq x_1$ and $B + 1 < y_1$, since the solution is increasing with respect to y and decreasing with respect to x then $B + 1 < y_1 < y_2 < y_3 < ...$ and $A + 1 \geq x_1 > x_2 > x_3 > ...$, so the solution has an infinite positive semi-cycle with respect to y and an infinite negative semi-cycle with respect to x.

- Case (2): if $A+1 < x_1$ and $B+1 < y_1$, then we can conclude that the solution has an infinite positive semi-cycle with respect to y. As for x, we claim that the semi-cycle consist of a x_1 ends with x_i such that $1 \leq i \leq k+1$. If $i = k+2$, then
$$x_{k+2} = A + \frac{y_{k+1}}{y_1} > A + 1$$
imply that the solution of (2.0.1) has an infinite positive semi-cycle with respect to y and infinite positive semi-cycle with respect to x.

- Case (3): if $A+1 \geq x_1$ and $B+1 \geq y_1$, then we can conclude that the solution has an infinite negative semi-cycle with respect to x. As for y, we claim that the semi-cycle consist of a y_1 ends with y_i such that $1 \leq i \leq k+1$. If $i = k+2$, then
$$y_{k+2} = B + \frac{x_{k+1}}{x_1} < B + 1$$
imply that the solution of (2.0.1) has an infinite negative semi-cycle with respect to y and infinite negative semi-cycle with respect to x.

- Case (4): if $A + 1 < x_1$ and $B + 1 \geq y_1$, then we claim that the semi-cycle consist of a (x_1, y_1) ends with (x_i, y_i) such that $1 \leq i \leq k + 1$. If $i = k + 2$, then
$$x_{k+2} = A + \frac{y_{k+1}}{y_1} > A + 1 \text{ and } y_{k+2} = B + \frac{x_{k+1}}{x_1} < B + 1$$
mean that the solution of (2.0.1) has an infinite negative semi-cycle with respect to y and infinite positive semi-cycle with respect to x. Hence, any increasing solution with respect to x and decreasing with respect to y to system (2.0.1) is non-oscillatory positive with respect to x and negative with respect to y.

$\blacksquare$

Theorem 2.7. *System (2.0.1) has no non-oscillatory solutions.*

Proof. Assume that system (2.0.1) has a non-oscillatory solution, say $\{x_n, y_n\}_{n=-k}^{\infty}$ which has an infinite negative semi-cycle, and assume this semi-cycle starts with (x_N, y_N) satisfies $x_n \geq A+1$ and $y_n < B+1$ or $x_n < A+1$ and $y_n \geq B+1$ for all $n \geq N$. Then
Case (1):

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} \geq A+1 \text{ implies } y_n \geq y_{n-k} \text{ for } n \geq \max\{1, N-1\}$$

and

$$y_{n+1} = B + \frac{x_n}{x_{n-k}} < B+1 \text{ implies } x_n < x_{n-k} \text{ for } n \geq \max\{1, N-1\}$$

so for all $n \geq \max\{1, N\}$

$$x_{n-k} > x_n > x_{n+k} > ... \geq A+1$$

and

$$B+1 > y_{n+k} \geq y_n \geq y_{n-k} \geq ... \geq B$$

implies the solution is bounded, which means that $\{x_n\}, \{y_n\}$ have k subsequences $\{x_{nk}\}, \{x_{nk+1}, ..., \{x_{nk+(k-1)}\}$ and $\{y_{nk}\}, \{y_{nk+1}, ..., \{y_{nk+(k-1)}\}$ such that each subsequence of $\{x_n\}$ is decreasing and bounded from below and each subsequence of $\{y_n\}$ is increasing and bounded from above, so each one of all subsequences is convergent, so for all $i = 0, 1, ..., k-1$ there exist γ_i, δ_i such that

$$\lim_{n\to\infty} x_{nk+i} = \gamma_i \text{ and } \lim_{n\to\infty} y_{nk+i} = \delta_i$$

Thus,

$$(\gamma_0, \delta_0), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$$

is a k-periodic solution of system (2.0.1), which contradicts the previous theorem (2.2) until the solution is the trivial solution. Hence, the solution converges to the equilibrium, which is a contradiction, due to the solution is diverging from the equilibrium. Hence, system (2.0.1) has no non-oscillatory solutions which have positive semi-cycles with respect to x and negative semi-cycles with respect to y (or negative with respect to x and positive with respect to y).

Case (2):

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} < A + 1 \text{ implies } y_n < y_{n-k} \text{ for } n \geq \max\{1, N-1\}$$

and

$$y_{n+1} = B + \frac{x_n}{x_{n-k}} \geq B + 1 \text{ implies } x_n \geq x_{n-k} \text{ for } n \geq \max\{1, N-1\}$$

so for $n \geq \max\{1, N\}$

$$A + 1 > x_{n+k} \geq x_n \geq x_{n-k} \geq ... \geq A$$

and

$$y_{n-k} > y_n > y_{n+k} > ... \geq B + 1$$

implies the solution is bounded, which means that $\{x_n\}, \{y_n\}$ have k subsequences $\{x_{nk}\}, \{x_{nk+1}, ..., \{x_{nk+(k-1)}\}$ and $\{y_{nk}\}, \{y_{nk+1}, ..., \{y_{nk+(k-1)}\}$ such that each subsequence of $\{x_n\}$ is increasing and bounded from above and each subsequence of $\{y_n\}$ is decreasing and bounded from below, so each one of all subsequences is convergent, so for all $i = 0, 1, ..., k-1$ there exist γ_i, δ_i such that

$$\lim_{n\to\infty} x_{nk+i} = \gamma_i \text{ and } \lim_{n\to\infty} y_{nk+i} = \delta_i$$

Thus,

$$(\gamma_0, \delta_0), (\gamma_1, \delta_1), ..., (\gamma_{k-1}, \delta_{k-1})$$

is a k-periodic solution of system (2.0.1), which contradicts the previous theorem (2.2) until the solution is the trivial solution. Hence, the solution converges to the equilibrium, which is a contradiction, due to the solution is diverging from the equilibrium. Hence, system (2.0.1) has no non-oscillatory solutions which have positive semi-cycles with respect to x and negative semi-cycles with respect to y (or negative with respect to x and positive with respect to y). ■

Corollary 2.7.1. *If $A \geq 1$ and $B \geq 1$, then system (2.0.1) has no increasing (resp. decreasing) solution with respect to x and decreasing (resp. increasing) with respect to y.*

Proof. Assume that system (2.0.1) has an increasing solution with respect to x and decreasing with respect to y, or a decreasing solution with respect to x and increasing with respect to y. Then from theorem (2.6). the solution is non-oscillatory and departs from the equilibrium $(A + 1, B + 1)$ which contradicts theorem (2.7) and theorem (2.3) ■

2.3 The Case $0 < A < 1$ and $0 < B < 1$

In this section, we study the asymptotic behavior of system (2.0.1) when $0 < A < 1$ and $0 < B < 1$. System (2.0.1) can have unbounded solutions given specific conditions.

Theorem 2.8. *Suppose that $0 < A < 1$ and $0 < B < 1$. Let $c = \max\{A, B\}$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ be an arbitrary positive solution of (2.0.1). Then the following statements are true:*

(a) If k is odd and $0 < x_{2m-1} < 1, x_{2m} > \frac{1}{1-c}, y_{2m-1} > \frac{1}{1-c}, 0 < y_{2m} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, ..., 0$, then $\lim_{n\to\infty} x_{2n} = \infty$, $\lim_{n\to\infty} y_{2n+1} = \infty$, $\lim_{n\to\infty} x_{2n+1} = A$, $\lim_{n\to\infty} y_{2n} = B$

(b) If k is odd and $0 < x_{2m} < 1, x_{2m-1} > \frac{1}{1-c}, y_{2m} > \frac{1}{1-c}, 0 < y_{2m-1} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, ..., 0$, then $\lim_{n\to\infty} x_{2n+1} = \infty$, $\lim_{n\to\infty} y_{2n} = \infty$, $\lim_{n\to\infty} x_{2n} = A$, $\lim_{n\to\infty} y_{2n+1} = B$

Proof.

- If k is odd and $0 < x_{2m-1} < 1, x_{2m} > \frac{1}{1-c}, y_{2m-1} > \frac{1}{1-c}, 0 < y_{2m} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, ..., 0$, then

$$0 < x_1 = A + \frac{y_0}{y_{-k}} < A + \frac{1}{y_{-k}} < A + 1 - c \leq A + 1 - A = 1$$

$$y_1 = B + \frac{x_0}{x_{-k}} > B + x_0 > x_0 > \frac{1}{1-c}$$

$$x_2 = A + \frac{y_1}{y_{-k+1}} > A + y_1 > y_1 > \frac{1}{1-c}$$

$$0 < y_2 = B + \frac{x_1}{x_{-k+1}} < B + \frac{1}{x_{-k+1}} < B + 1 - c \leq B + 1 - B = 1$$

By induction, we get that for $n = 1, 2, ...$

$$0 < x_{2n-1} < 1, x_{2n} > \frac{1}{1-c}, y_{2n-1} > \frac{1}{1-c}, 0 < y_{2n} < 1$$

so for $l \geq 1$

$$x_{2l} = A + \frac{y_{2l-1}}{y_{2l-(k+1)}} > A + y_{2l-1} = A + B + \frac{x_{2l-2}}{x_{2l-k-2}} > A + B + x_{2l-2}$$

$$x_{4l} = A + \frac{y_{4l-1}}{y_{4l-(k+1)}} > A + y_{4l-1} = A + B + \frac{x_{4l-2}}{x_{4l-k-2}} > A + B + x_{4l-2}$$

$$= 2A + B + \frac{y_{4l-3}}{y_{4l-k-3}} > 2A + B + y_{4l-3} = 2A + 2B + \frac{x_{4l-4}}{x_{4l-k-4}} > 2A + 2B + x_{4l-4}$$

also

$$x_{6l} > 3A + 3B + x_{6l-6}$$

so for all $r = 1, 2, \ldots$

$$x_{2rl} > r(A + B) + x_{2rl-2r}$$

if $n = rl$, then as $r \to \infty$ and $\lim_{n\to\infty} x_{2n} = \infty$. Considering (2.0.1) and taking the limit on both sides of the equation

$$y_{2n+1} = B + \frac{x_{2n}}{x_{2n-k}}$$

we get $\lim_{n\to\infty} y_{2n+1} = \infty$ since $0 < x_{2n-k} < 1$ for all $n = 0, 1, \ldots$ Now, take the limit on both sides of the equation

$$x_{2n+1} = A + \frac{y_{2n}}{y_{2n-k}}$$

we obtain $\lim_{n\to\infty} x_{2n+1} = A$ since $0 < y_{2n} < 1$ for all n. Now, take the limit on both sides of the equation

$$y_{2n+2} = B + \frac{x_{2n+1}}{x_{2n-k+1}}$$

to get $\lim_{n\to\infty} y_{2n} = B$, which completes the proof of (a)

- If k is odd and $0 < x_{2m} < 1, x_{2m-1} > \frac{1}{1-c}, y_{2m} > \frac{1}{1-c}, 0 < y_{2m-1} < 1$ for $m = \frac{1-k}{2}, \frac{3-k}{2}, \ldots, 0$, then

$$x_1 = A + \frac{y_0}{y_{-k}} > A + y_0 > y_0 > \frac{1}{1-c}$$

$$0 < y_1 = B + \frac{x_0}{x_{-k}} < B + \frac{1}{x_{-k}} < B + 1 - c \leq B + 1 - B = 1$$

$$0 < x_2 = A + \frac{y_1}{y_{-k+1}} < A + \frac{1}{y_{-k+1}} < A + 1 - c \leq A + 1 - A = 1$$

$$y_2 = B + \frac{x_1}{x_{-k+1}} > B + x_1 > x_1 > \frac{1}{1-c}$$

By induction, we have for all $n = 1, 2, ...$

$$0 < x_{2n} < 1, x_{2n-1} > \frac{1}{1-c}, y_{2n} > \frac{1}{1-c}, 0 < y_{2n-1} < 1$$

so for $l \geq 1$

$$x_{2l+1} = A + \frac{y_{2l}}{y_{2l-k}} > A + y_{2l} = A + B + \frac{x_{2l-1}}{x_{2l-k-1}} > A + B + x_{2l-1}$$

$$x_{4l+1} = A + \frac{y_{4l}}{y_{4l-k}} > A + y_{4l} = A + B + \frac{x_{4l-1}}{x_{4l-k-1}} > A + B + x_{4l-1}$$

$$= 2A + B + \frac{y_{4l-2}}{y_{4l-k-2}} > 2A + B + y_{4l-2} > 2A + 2B + \frac{x_{4l-3}}{x_{4l-k-3}} > 2A + 2B + x_{4l-3}$$

also, $x_{6l+1} > 3A + 3B + x_{6l-5}$. So for all $r = 1, 2, ...$

$$x_{2rl+1} > r(A + B) + x_{2rl-(2r-1)}$$

if $n = rl$, then as $r \to \infty, n \to \infty$ and $\lim_{n\to\infty} x_{2n+1} = \infty$. Considering (2.0.1) and taking the limit on both sides of the equation

$$y_{2n+2} = B + \frac{x_{2n+1}}{x_{2n-k+1}}$$

we get $\lim_{n\to\infty} y_{2n} = \infty$ since $0 < x_{2n-k+1} < 1$ for all $n = 0, 1, ...$ Now, take the limit on both sides of the equation

$$y_{2n+1} = B + \frac{x_{2n}}{x_{2n-k}}$$

we obtain $\lim_{n\to\infty} y_{2n+1} = B$ since $0 < x_{2n} < 1$ for all n. Now, take the limit on both sides of the equation

$$x_{2n+2} = A + \frac{y_{2n+1}}{y_{2n-k+1}}$$

to get $\lim_{n\to\infty} x_{2n} = A$, which completes the proof.

■

2.4 The Case $A > 1$ and $B > 1$

In this section, we study the boundedness and persistence of the positive solutions of system (2.0.1) when $A > 1$ and $B > 1$. We also prove that if $A > 1$ and $B > 1$ then the unique positive equilibrium of (2.0.1) is globally asymptotically stable.

Lemma 2.9. *Given v_j, where $j = -k, -k+1, ..., k+1$. Then the solution of the second order linear difference equation*

$$v_{n+2} = av_n + b, n \geq k, a \neq 1$$

is of the form

$$v_{k+2l} = \left(v_k + \frac{b}{a-1}\right)a^l + \frac{b}{1-a}$$

$$v_{k+2l+1} = \left(v_{k+1} + \frac{b}{a-1}\right)a^l + \frac{b}{1-a}$$

for all $l \geq 0$

Proof.

$$\begin{aligned}
v_{k+2} &= av_k + b, \\
v_{k+3} &= av_{k+1} + b, \\
v_{k+4} &= av_{k+2} + b = a^2 v_k + ab + b, \\
v_{k+5} &= av_{k+3} + b = a^2 v_{k+1} + ab + b, \\
v_{k+6} &= av_{k+4} + b = a^3 v_k + a^2 b + ab + b, \\
v_{k+7} &= av_{k+5} + b = a^3 v_{k+1} + a^2 b + ab + b,
\end{aligned}$$

hence, for all $l \geq 0$

$$v_{k+2l} = a^1 v_k + b(a^{l-1} + a^{l-2} + ... + 1) = \left(v_k + \frac{b}{a-1}\right)a^l + \frac{b}{1-a}$$

$$v_{k+2l+1} = a^1 v_{k+1} + b(a^{l-1} + a^{l-2} + ... + 1) = \left(v_{k+1} + \frac{b}{a-1}\right)a^l + \frac{b}{1-a}$$

which completes the proof. ■

Theorem 2.10. *Suppose that $A > 1$ and $B > 1$. Then every positive solution of system (2.0.1) is bounded and persists. In fact, for all $l \geq 0$,*

$$A < x_{k+2l} \leq \left(x_k + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1}$$

and

$$A < x_{k+2l+1} \leq \left(x_{k+1} + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1}$$

similarly,

$$B < y_{k+2l} \leq \left(y_k + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1}$$

and

$$B < y_{k+2l+1} \leq \left(y_{k+1} + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1}$$

Proof. Assume $A > 1$, $B > 1$ and $\{x_n, y_n\}_{n=-k}^{\infty}$ is a positive solution of system (2.0.1). Since $x_n > 0$ and $y_n > 0$ for all $n \geq -k$, (2.0.1) implies that

$$x_n > A > 1, y_n > B > 1 \text{ for all } n \geq 1 \tag{2.4.1}$$

Now, using (2.0.1) and (2.4.1) we get that for all $n \geq 2$

$$\begin{aligned} x_n &= A + \frac{y_{n-1}}{y_{n-k-1}} < A + \frac{1}{B}y_{n-1} \\ y_n &= B + \frac{x_{n-1}}{x_{n-k-1}} < B + \frac{1}{A}x_{n-1} \end{aligned} \tag{2.4.2}$$

Let v_n, w_n be the solution of the system

$$v_n = A + \frac{1}{B}w_{n-1}, w_n = B + \frac{1}{A}v_{n-1}, \text{ for all } n \geq k+1 \tag{2.4.3}$$

such that

$$v_i = x_i, w_i = y_i, i = 1, 2, ..., k + 1 \tag{2.4.4}$$

now, we use induction to prove that

$$x_n < v_n, \; y_n < w_n, \text{ for all } n \geq k + 2 \tag{2.4.5}$$

Suppose that (2.4.5) is true for $n = m \geq k + 2$. Then from (2.4.2), we get

$$\begin{aligned} x_{m+1} &< A + \frac{1}{B} y_m < A + \frac{1}{B} w_m = v_{m+1} \\ y_{m+1} &< B + \frac{1}{A} x_m < B + \frac{1}{A} v_m = w_{m+1} \end{aligned} \tag{2.4.6}$$

Therefore, (2.4.5)is true. From (2.4.3) and (2.4.4), we have

$$v_{n+2} = A + 1 + \frac{1}{AB} v_n, \tag{2.4.7}$$

$$w_{n+2} = B + 1 + \frac{1}{AB} w_n, \tag{2.4.8}$$

$$n \geq k \tag{2.4.9}$$

for simplicity, let $a = \frac{1}{AB}$, $b = A + 1$ and $c = B + 1$. Then (2.4.7) becomes

$$v_{n+2} = a v_n + b, \; w_{n+2} = a w_n + c, \; n \geq k$$

Now, using Lemma(2.9), for all $l \geq 0$

$$v_{k+2l} = a^l x_k + b(a^{l-1} + a^{l-2} + ... + 1) = \left(x_k + \frac{b}{a-1} \right) a^l + \frac{b}{1-a}$$

$$v_{k+2l+1} = a^l x_{k+1} + b(a^{l-1} + a^{l-2} + ... + 1) = \left(x_{k+1} + \frac{b}{a-1} \right) a^l + \frac{b}{1-a}$$

since $A > 1, B > 1$ and $a = \frac{1}{AB}, b = A + 1$. Then for all $l \geq 0$

$$v_{k+2l} = \left(x_k + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1}$$

$$v_{k+2l+1} = \left(x_{k+1} + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1} \tag{2.4.10}$$

Then, from (2.4.1), (2.0.1) and(2.4.10), for all $l \geq 0$

$$A < x_{k+2l} \leq \left(x_k + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1}$$

$$A < x_{k+2l+1} \leq \left(x_{k+1} + \frac{(A+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(A+1)AB}{AB-1}$$

And since $A > 1, B > 1$ and $a = \frac{1}{AB}, b = A + 1$. Then for all $l \geq 0$

$$w_{k+2l} = \left(y_k + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1}$$

$$w_{k+2l+1} = \left(y_{k+1} + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1} \tag{2.4.11}$$

Then, from (2.4.1), (2.0.1) and(2.4.11), for all $l \geq 0$

$$B < y_{k+2l} \leq \left(y_k + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1}$$

$$B < y_{k+2l+1} \leq \left(y_{k+1} + \frac{(B+1)AB}{1-AB}\right)\left(\frac{1}{AB}\right)^l + \frac{(B+1)AB}{AB-1}$$

The proof is complete ■

Theorem 2.11. *If $A > 1$ and $B > 1$. Then every positive solution of system (2.0.1) converges to the equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ as $n \to \infty$.*

Proof. Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be an arbitrary positive solution of (2.0.1), and let

$$u_1 = \lim_{n\to\infty} \sup x_n, \quad l_1 = \lim_{n\to\infty} \inf x_n$$

$$u_2 = \lim_{n\to\infty} \sup y_n, \quad l_2 = \lim_{n\to\infty} \inf y_n$$

Now, system (2.0.1) implies that

$$u_1 \le A + \frac{u_2}{l_2}, u_2 \le B + \frac{u_1}{l_1}, l_1 \ge A + \frac{l_2}{u_2}, l_2 \ge B + \frac{l_1}{u_1} \tag{2.4.12}$$

from (2.4.12) we get $u_1 l_2 \le A l_2 + u_2$ and $u_1 l_2 \ge u_1 B + l_1$
then

$$Bu_1 + l_1 \le u_1 l_2 \le A l_2 + u_2 \tag{2.4.13}$$

$$Au_2 + l_2 \le u_2 l_1 \le B l_1 + u_1 \tag{2.4.14}$$

from (2.4.13) we get

$$Bu_1 + l_1 \le A l_2 + u_2 \tag{2.4.15}$$

and (2.4.14) implies

$$-Bl_1 - u_1 \le -Au_2 - l_2 \tag{2.4.16}$$

from (2.4.15) and (2.4.16) we get

$$Bu_1 + l_1 - Bl_1 - u_1 \le Al_2 + u_2 - Au_2 - l_2$$

and

$$(B-1)(u_1 - l_1) + (A-1)(u_2 - l_2) \le 0$$

but $A > 1$ and $B > 1$ so $A - 1 > 0$ and $B - 1 > 0$, also $u_1 - l_1, u_2 - l_2 \geq 0$. Hence

$$u_1 - l_1 = 0 \text{ and } u_2 - l_2 = 0$$

so $u_1 = l_1$ and $u_2 = l_2$. Now use (2.4.12) to get

$$B + 1 \leq l_2 = u_2 \leq B + 1 \text{ and } A + 1 \leq l_1 = u_1 \leq A + 1$$

hence

$$l_1 = u_1 = A + 1 \text{ and } l_2 = u_2 = B + 1$$

so

$$\lim_{n\to\infty} x_n = l_1 = u_1 = A + 1 \text{ and } \lim_{n\to\infty} y_n = l_2 = u_2 = B + 1$$

which completes the proof. ■

Lemma 2.12. *If $A > 1$ and $0 < \epsilon < \frac{A-1}{(A+1)(k+1)}$ where $k \in Z^+$, then $\frac{2}{(1-(k+1)\epsilon)(A+1)} < 1$.*

Proof.

$$0 < \epsilon < \frac{1}{(k+1)}\frac{A-1}{A+1} \text{ implies } 0 < (k+1)\epsilon < \frac{A-1}{A+1}$$

so

$$1 - (k+1)\epsilon > 1 - \frac{A-1}{A+1} = \frac{2}{A+1}$$

that is

$$\frac{1}{1-(k+1)\epsilon} < \frac{A+1}{2} \text{ implies } \frac{2}{1-(k+1)\epsilon} < A + 1$$

and so

$$\frac{2}{(1-(k+1)\epsilon)(A+1)} < 1$$

The proof is complete. ■

Theorem 2.13. *If $A > 1$ and $B > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.0.1) is locally asymptotically stable.*

Proof. System (2.0.1) can be formulated as a system of first order recurrence equations as follows:

$$\begin{aligned} w_n^1 = x_n, w_n^2 = x_{n-1}, ..., w_n^{(k+1)} = x_{n-k} \\ v_n^1 = y_n, v_n^2 = y_{n-1}, ..., v_n^{(k+1)} = y_{n-k} \end{aligned} \tag{2.4.17}$$

Let $Z_n = (w_n^1, w_n^2, ..., w_n^{(k+1)}, v_n^1, v_n^2, ..., v_n^{(k+1)})^T$. Then the linearized system of system (2.0.1) associated with (2.4.17) about the equilibrium point $(\bar{x}, \bar{y}) = (A+1, B+1)$ is

$$Z_{n+1} = JZ_n$$

where

$$Z_{n+1} = \begin{pmatrix} w_{n+1}^{(1)} \\ w_{n+1}^{(2)} \\ \vdots \\ w_{n+1}^{(k+1)} \\ v_{n+1}^{(1)} \\ v_{n+1}^{(2)} \\ \vdots \\ v_{n+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} A + \frac{v_n^1}{v_n^{(k+1)}} \\ w_n^{(1)} \\ \vdots \\ w_n^{(k)} \\ B + \frac{w_n^1}{w_n^{(k+1)}} \\ v_n^{(1)} \\ \vdots \\ v_n^{(k)} \end{pmatrix}$$

and J is the Jacobian matrix.

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} D_{w_n^{(1)}} Z_{n+1} & ... & D_{w_n^{(k+1)}} Z_{n+1} & D_{v_n^{(1)}} Z_{n+1} & ... & D_{v_n^{(k+1)}} Z_{n+1} \end{pmatrix}$$

so the Jacobian matrix will be of the following form

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{B+1} & 0 & \cdots & 0 & \frac{-1}{B+1} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{A+1} & 0 & \cdots & 0 & \frac{-1}{A+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Let $\lambda_1, \lambda_2, ..., \lambda_{2k+2}$ be the eigenvalues of J. Define $D = diag(d_1, d_2, ..., d_{2k+2})$ be a diagonal matrix such that

$$d_1 = d_{k+2} = 1, \quad d_m = d_{k+1+m} = 1 - m\epsilon, \quad m = 2, 3, ..., k+1$$

choose $\epsilon > 0$ such that $0 < \epsilon < \min\{\frac{A-1}{(A+1)(k+1)}, \frac{B-1}{(B+1)(k+1)}\}$. Now,

$$D_{(2k+2)\times(2k+2)} = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & d_{2k+2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-2\epsilon & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1-(k+1)\epsilon & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1-2\epsilon & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1-(k+1)\epsilon \end{pmatrix}$$

so for all $m = 2, 3, ..., k+1$, by Lemma (2.12)

$$1 - m\epsilon \geq 1 - (k+1)\epsilon > 1 - \frac{(k+1)(A-1)}{(k+1)(A+1)} = \frac{A+1-A+1}{A+1} = \frac{2}{A+1} > 0$$

so for all $m, 1 - m\epsilon > 0$, hence D is invertible. Now,

$$DJD^{-1}_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{B+1}\frac{d_1}{d_{k+2}} & 0 & \cdots & 0 & \frac{-1}{B+1}\frac{d_1}{d_{2k+2}} \\ \frac{d_2}{d_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{d_{k+1}}{d_k} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{A+1}\frac{d_{k+2}}{d_1} & 0 & \cdots & 0 & \frac{-1}{A+1}\frac{d_{k+2}}{d_{k+1}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{d_{2k+2}}{d_{2k+1}} & 0 \end{pmatrix}$$

Now, we need to show that the sum of the absolute value of entries of every row is much less than one, on the way to find the infinite norm of DJD^{-1}. Since $\epsilon > 0$ so $1 - m\epsilon > 1 - (m+1)\epsilon$, that is, $d_m > d_{m+1}$, for all m. So

$$\frac{d_2}{d_1} < 1, \frac{d_3}{d_2} < 1, ..., \frac{d_{2k+2}}{d_{2k+1}} < 1$$

$$\begin{aligned} For\ \tfrac{1}{B+1}\tfrac{d_1}{d_{k+2}} + \tfrac{1}{B+1}\tfrac{d_1}{d_{2k+2}} &= \tfrac{1}{B+1} + \tfrac{1}{(1-(k+1)\epsilon)(B+1)} \\ &= \tfrac{1}{B+1} + \tfrac{1}{(1-(k+1)\epsilon)(B+1)} \\ &< \tfrac{1}{1-(k+1)\epsilon}\tfrac{1}{B+1} + \tfrac{1}{(1-(k+1)\epsilon)\frac{1}{(}B+1)} \\ &< \tfrac{2}{(1-(k+1)\epsilon)(B+1)} \\ &< 1 \end{aligned}$$

$$\begin{aligned} For\ \tfrac{1}{A+1}\tfrac{d_{k+2}}{d_1} + \tfrac{1}{A+1}\tfrac{d_{k+2}}{d_{k+1}} &= \tfrac{1}{A+1} + \tfrac{1}{(1-(k+1)\epsilon)(A+1)} \\ &< \tfrac{1}{1-(k+1)\epsilon}\tfrac{1}{A+1} + \tfrac{1}{(1-(k+1)\epsilon)\frac{1}{(}A+1)} \\ &< \tfrac{2}{(1-(k+1)\epsilon)(A+1)} \\ &< 1 \end{aligned}$$

Since J has the same eigenvalue as DJD^{-1}. Then,

$$\rho(J) = max\{\mid \lambda_i \mid\} \leq \|DJD^{-1}\|_\infty$$

but

$$||DJD^{-1}||_\infty = \left\{ \begin{matrix} \frac{1}{B+1} + \frac{1}{(1-(1+k)\epsilon)(B+1)}, \frac{d2}{d1}, \frac{d3}{d2}, \ldots, \frac{d_{k+1}}{d_k}, \\ \frac{1}{A+1} + \frac{1}{(1-(1+k)\epsilon)(A+1)} \end{matrix} \right\} < 1.$$

So the modulus of each eigenvalue of J is much less than one. Hence, the unique equilibrium point $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.0.1) is locally asymptotically stable. ∎

Theorem 2.14. *If $A > 1$ and $B > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.0.1) is globally asymptotically stable.*

Proof. Using theorem (2.13) we conclude that the equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.0.1) is locally asymptotically stable, but Theorem (2.11) implies that this equilibrium is a global attractor. Thus, the unique positive equilibrium $(\bar{x}, \bar{y}) = (A+1, B+1)$ of system (2.0.1) is globally asymptotically stable. ∎

2.5 Numerical Examples

In this section, we give some numerical examples that represent different cases of dynamical behavior of solutions of (2.0.1) the use of MATLAB to illustrate the results we had in the previous sections.

Example 2.1. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-5}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-5}}, \quad n = 0, 1, \cdots \tag{2.5.1}$$

with $A = 0.2, B = 0.8$, and the initial conditions $x_{-5} = 0.7, x_{-4} = 9.1, x_{-3} = 0.2, x_{-2} = 9.2, x_{-1} = 0.3, x_0 = 10, y_{-5} = 0.4, y_{-4} = 10.3, y_{-3} = 0.5, y_{-2} = 9.3, y_{-1} = 0.3, y_0 = 11.2$. Then the solution of system (2.5.1) is unbounded since $0 < A < 1$ and $0 < B < 1$ and the initial conditions in Theorem (2.8) are satisfying. The unique positive equilibrium point $(\bar{x}, \bar{y}) = (1.2, 1.8)$ is not globally asymptotically stable (see Figure 1.1, Theorem (2.8)).

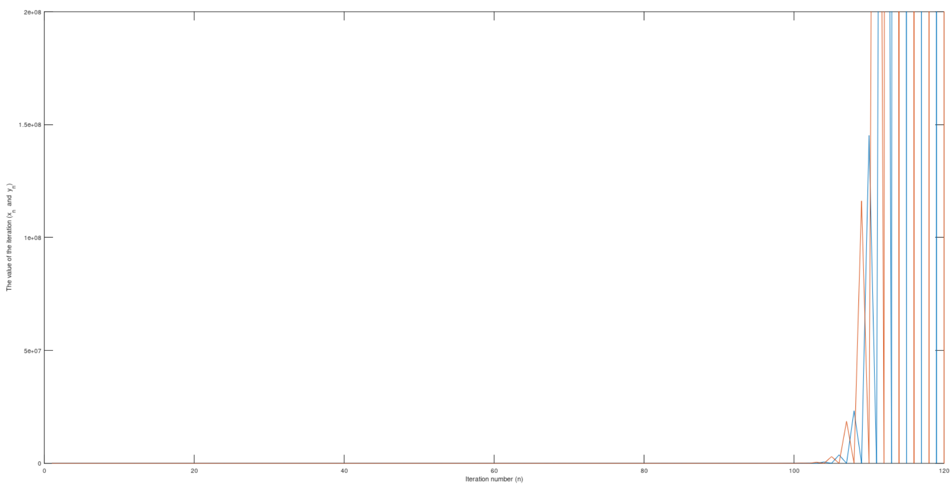

Fig. 2.1: The graph of a solution of system (2.5.1) with $A = 0.2$ and $B = 0.8$

Example 2.2. Consider system (2.5.1) with $A = 4, B = 2.5$, and the initial conditions $x_{-5} = 2.5, x_{-4} = 3.7, x_{-3} = 1.5, x_{-2} = 0.7, x_{-1} = 0.3, x_0 = 0.4, y_{-5} = 2.2, y_{-4} = 3.3, y_{-3} = 1.2, y_{-2} = 0.3, y_{-1} = 0.2, y_0 = 0.9$. Since $A > 1$ and $B > 1$, the solution of system (2.5.1) is bounded and persists (see Theorem (2.9)), and the unique positive equilibrium $(\bar{x}, \bar{y}) = (5, 3.5)$ is globally asymptotically stable (see Figure 1.2, Theorem (2.14)).

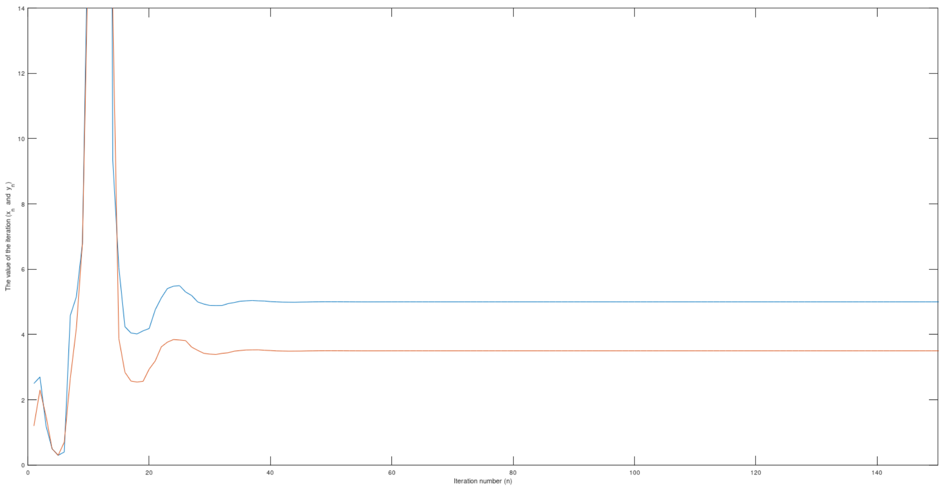

Fig. 2.2: The graph of a solution of system (2.5.1) with $A = 4$ and $B = 2.5$

Example 2.3. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{y_n}{y_{n-4}}, \quad y_{n+1} = B + \frac{x_n}{x_{n-4}}, \quad n = 0, 1, \cdots \tag{2.5.2}$$

with $A = 3, B = 4$, and the initial conditions $x_{-4} = 0.8, x_{-3} = 1.3, x_{-2} = 1.2, x_{-1} = 2.1, x_0 = 1.2, y_{-4} = 1.5, y_{-3} = 2.3, y_{-2} = 0.3, y_{-1} = 0.5, y_0 = 0.7$. Then the solution of system (2.5.2) Then the unique positive equilibrium $(\bar{x}, \bar{y}) = (4, 5)$ is globally asymptotically stable since $A > 1$ and $B > 1$ (see Theorem (2.14)), and the solution of system (2.5.2)is bounded and persists (see Figure 1.3, Theorem (2.9)). In this example $k = 4$ is even, while in Example 1.2, $k = 5$ is odd, but in both cases we have the same conclusion.

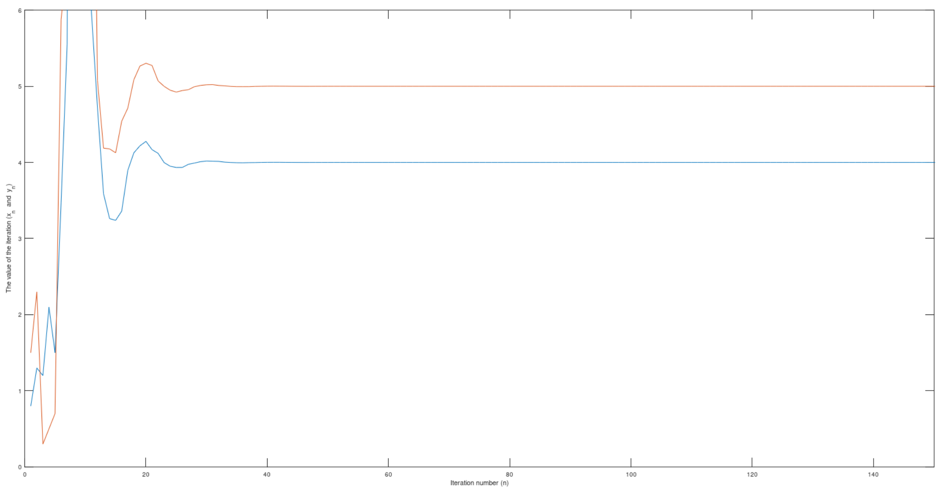

Fig. 2.3: The graph of a solution of system (2.5.2) with $A = 3$ and $B = 4$

3. DYNAMICS OF THE SYSTEM $X_{N+1} = A + \frac{X_N}{Y_{N-K}}, \quad Y_{N+1} = B + \frac{Y_N}{X_{N-K}}$

In this chapter, we introduce the dynamical system:

$$x_{n+1} = A + \frac{x_n}{y_{n-k}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-k}}, \quad n = 0, 1, \cdots \tag{3.0.1}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$. We study the dynamical behavior of this system in the cases: case (1): $0 < A < 1$, $0 < B < 1$ case (2): $A > 1$, $B > 1$. Moreover we also investigate the behavior of the positive solutions of (3.0.1) using the semi-cycle analysis method. Finally, we give some numerical examples that illustrate the results in this chapter.

System (3.0.1) has the unique positive equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ since $\bar{x} = A + \frac{\bar{x}}{\bar{y}}, \quad \bar{y} = B + \frac{\bar{y}}{\bar{x}}$ implies that

$$\bar{x}\bar{y} = A\bar{y} + \bar{x}, \quad \bar{x}\bar{y} = B\bar{x} + \bar{y}$$

so

$$A\bar{y} + \bar{x} = B\bar{x} + \bar{y}$$

$$A\bar{y} - \bar{y} = B\bar{x} - \bar{x}$$

$$(A-1)\bar{y} = (B-1)\bar{x}$$

hence, we find

$$\bar{y} = \frac{B-1}{A-1}\bar{x}$$

so

$$\frac{\bar{x}}{\bar{y}} = \frac{A-1}{B-1}$$

and

$$\bar{x} = A + \frac{\bar{x}}{\bar{y}} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1},$$
$$\bar{y} = B + \frac{\bar{y}}{\bar{x}} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$
$$(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1}).$$

3.1 Semi-cycle Analysis I

In this section, we examine the behavior of positive solutions of system (3.0.1) via semi-cycle analysis method.

Theorem 3.1. *Let $\{x_n, y_n\}_{n=-k}^{\infty}$ be a solution of system (3.0.1). Then, either this solution is non-oscillatory solution or it oscillates about the equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ with semi-cycles such that if there exists a semi-cycle with at least k terms, then every semi-cycle after that has at least $k+1$ terms.*

Proof. Assume $\{x_n, y_n\}_{n=-k}^{\infty}$ is a solution of system (3.0.1), and there exists an integer $n_0 \geq 0$ such that (x_{n_0}, y_{n_0}) is the last term of a semi-cycle that has at least k terms. Then, either

$$..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} < \frac{AB-1}{B-1} \leq x_{n_0+1}$$

and

$$..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} < \frac{AB-1}{A-1} \leq y_{n_0+1}$$

or

$$..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} \geq \frac{AB-1}{B-1} > x_{n_0+1}$$

and

$$..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} \geq \frac{AB-1}{A-1} > y_{n_0+1}$$

- Case 1: If $..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} < \frac{AB-1}{B-1} \leq x_{n_0+1}$ and $..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} < \frac{AB-1}{A-1} \leq y_{n_0+1}$, then

$$x_{n_0+2} = A + \frac{x_{n_0+1}}{y_{n_0-k+1}} > A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+2} = B + \frac{y_{n_0+1}}{x_{n_0-k+1}} > B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

$$x_{n_0+3} = A + \frac{x_{n_0+2}}{y_{n_0-k+2}} > A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+3} = B + \frac{y_{n_0+2}}{x_{n_0-k+2}} > B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

$$\vdots$$

$$x_{n_0+k} = A + \frac{x_{n_0+k-1}}{y_{n_0-1}} > A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+k} = B + \frac{y_{n_0+k-1}}{x_{n_0-1}} > B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

$$x_{n_0+k+1} = A + \frac{x_{n_0+k}}{y_{n_0}} > A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+k+1} = B + \frac{y_{n_0+k}}{x_{n_0}} > B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

hence, the semi-cycle beginning with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume the semi-cycle which begins with (x_{n_0+1}, y_{n_0+1}) has exactly $k+1$ terms, then the following semi-cycle will begin with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that

$$x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+k+1} \geq \frac{AB-1}{B-1} > x_{n_0+k+2}$$

and

$$y_{n_0+1}, y_{n_0+2}, ..., y_{n_0+k+1} \geq \frac{AB-1}{A-1} > y_{n_0+k+2},$$

then for $i = 1, 2, 3, .., k$

$$x_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{y_{n_0+1+i}} < A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+k+2+i} = B + \frac{y_{n_0+k+1+i}}{x_{n_0+1+i}} < B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

so, each semi-cycle after this point must have at least $k+1$ terms.

- Case 2: If $..., x_{n_0-k+1}, ..., x_{n_0-1}, x_{n_0} \geq \frac{AB-1}{B-1} > x_{n_0+1}$ and $..., y_{n_0-k+1}, ..., y_{n_0-1}, y_{n_0} \geq \frac{AB-1}{A-1} > y_{n_0+1}$, then for all $i = 2, 3, ..., k+1$

$$x_{n_0+i} = A + \frac{x_{n_0-1+i}}{y_{n_0-k-1+i}} < A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+i} = B + \frac{y_{n_0-1+i}}{x_{n_0-k-1+i}} < B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

hence, the semi-cycle beginning with (x_{n_0+1}, y_{n_0+1}) has at least $k+1$ terms. Now, assume this semi-cycle has exactly $k+1$ terms, then the following semi-cycle will begin with $(x_{n_0+k+2}, y_{n_0+k+2})$ such that

$$x_{n_0+1}, x_{n_0+2}, ..., x_{n_0+k+1} < \frac{AB-1}{B-1} \leq x_{n_0+k+2}$$

and

$$y_{n_0+1}, y_{n_0+2}, ..., y_{n_0+k+1} < \frac{AB-1}{A-1} \leq y_{n_0+k+2}$$

then for $i = 1, 2, ..., k$

$$x_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{y_{n_0+1+i}} > A + \frac{AB-1}{B-1}\frac{A-1}{AB-1} = A + \frac{A-1}{B-1} = \frac{AB-1}{B-1}$$

and

$$y_{n_0+k+2+i} = B + \frac{y_{n_0+k+1+i}}{x_{n_0+1+i}} > B + \frac{AB-1}{A-1}\frac{B-1}{AB-1} = B + \frac{B-1}{A-1} = \frac{AB-1}{A-1}$$

so, every semi-cycle after this point must have at least $k+1$ terms. The proof is complete.

■

3.2 The Case $0 < A < 1$ and $0 < B < 1$

In this section, we study the asymptotic behavior of the positive solutions of system (3.0.1) when $0 < A < 1$ and $0 < B < 1$. System (3.0.1) can have unbounded solutions given certain conditions.

Theorem 3.2. *Suppose that $0 < A < 1$ and $0 < B < 1$. Let $\{x_n, y_n\}$ be an arbitrary positive solution of (3.0.1). Then the following statements are true:*

(a) If $\frac{x_0}{y_{-k}} < 1 - A$, $\frac{y_0}{x_{-k}} > \frac{1}{1-A} - B$ and $y_{-k+1}, ..., y_0 > \frac{1}{1-A}$, $x_{-k+1}, ..., x_0 < 1$. Then $x_n < 1, y_n > \frac{1}{1-A}$ and $x_n \to A, y_n \to \infty$

(b) If $\frac{y_0}{x_{-k}} < 1 - B$, $\frac{x_0}{y_{-k}} > \frac{1}{1-B} - A$ and $y_{-k+1}, ..., y_0 < 1$ $x_{-k+1}, ..., x_0 > \frac{1}{1-B}$. Then $y_n < 1, x_n > \frac{1}{1-B}$ and $x_n \to \infty, y_n \to B$

Proof. • If $\frac{x_0}{y_{-k}} < 1-A$, $\frac{y_0}{x_{-k}} > \frac{1}{1-A} - B$ and $y_{-k+1}, ..., y_0 > \frac{1}{1-A}$, $x_{-k+1}, ..., x_0 < 1$. Then

$$x_1 = A + \frac{x_0}{y_{-k}} < A + 1 - A = 1$$

$$y_1 = B + \frac{y_0}{x_{-k}} > B + \frac{1}{1-A} - B = \frac{1}{1-A}$$

$$x_2 = A + \frac{x_1}{y_{-k+1}} < A + \frac{1}{y_{-k+1}} < A + 1 - A = 1$$

$$y_2 = B + \frac{y_1}{x_{-k+1}} > B + \frac{1}{1-A}\frac{1}{x_{-k+1}} > B + \frac{1}{1-A} > \frac{1}{1-A}$$

$$x_3 = A + \frac{x_2}{y_{-k+2}} < A + \frac{1}{y_{-k+2}} < A + 1 - A = 1$$

$$y_3 = B + \frac{y_2}{x_{-k+2}} > B + \frac{1}{1-A}\frac{1}{x_{-k+2}} > B + \frac{1}{1-A} > \frac{1}{1-A}$$

$$x_{k+1} = A + \frac{x_k}{y_0} < A + \frac{1}{y_0} < A + 1 - A = 1$$

$$y_{k+1} = B + \frac{y_k}{x_0} > B + \frac{1}{1-A}\frac{1}{x_0} > B + \frac{1}{1-A} > \frac{1}{1-A}$$

$\vdots$

$$y_{n+1} = B + \frac{y_n}{x_{n-k}} > B + y_n$$

implies $\lim_{n\to\infty} y_n = \infty$, $\lim_{n\to\infty} x_n = A$.

- If $\frac{y_0}{x_{-k}} < 1 - B$, $\frac{x_0}{y_{-k}} > \frac{1}{1-B} - A$ and $y_{-k+1}, ..., y_0 < 1$ $x_{-k+1}, ..., x_0 > \frac{1}{1-B}$. Then

$$x_1 = A + \frac{x_0}{y_{-k}} > A + \frac{1}{1-B} - A > \frac{1}{1-B}$$

$$y_1 = B + \frac{y_0}{x_{-k}} < B + 1 - B = 1$$

$$x_2 = A + \frac{x_1}{y_{-k+1}} > A + \frac{1}{1-B}\frac{1}{y_{-k+1}} > A + \frac{1}{1-B} > \frac{1}{1-B}$$

$$y_2 = B + \frac{y_1}{x_{-k+1}} < B + \frac{1}{x_{-k+1}} < B + 1 - B = 1$$

$$x_3 = A + \frac{x_2}{y_{-k+2}} > A + \frac{1}{1-B}\frac{1}{y_{-k+2}} > A + \frac{1}{1-B} > \frac{1}{1-B}$$

$$y_3 = B + \frac{y_2}{x_{-k+2}} < B + \frac{1}{x_{-k+2}} < B + 1 - B = 1$$

$$x_{k+1} = A + \frac{x_k}{y_0} < A + \frac{1}{1-B}\frac{1}{y_0} > A + \frac{1}{1-B} > \frac{1}{1-B}$$

$$y_{k+1} = B + \frac{y_k}{x_0} < B + \frac{1}{x_0} < B + 1 - B = 1$$

$\vdots$

$$x_{n+1} = A + \frac{x_n}{y_{n-k}} > A + x_n$$

implies $\lim_{n\to\infty} x_n = \infty$, $\lim_{n\to\infty} y_n = B$.

which completes the proof.

$\blacksquare$

3.3 The Case $A > 1$ and $B > 1$

In this section, we study the boundedness and persistence of the positive solutions of system (3.0.1) when $A > 1$ and $B > 1$, we additionally show that if $A > 1$ and $B > 1$ then the unique positive equilibrium of (3.0.1) is globally asymptotically stable.

Theorem 3.3. *[17] Suppose that $A > 1, B > 1$. Then each positive solution $\{x_n, y_n\}$ of (3.0.1) is bounded and persists. In particular, for $i = k+2, k+3, ..., 3k+3$ and $l \geq 0$, every positive solution of (3.0.1) satisfies*

$$\begin{aligned} A \leq x_{k+l} \leq (\frac{1}{B})^l \left(x_{k+1} - \frac{AB}{B-1} \right) + \frac{AB}{B-1}, \\ B \leq y_{k+l} \leq (\frac{1}{A})^l \left(y_{k+1} - \frac{AB}{A-1} \right) + \frac{AB}{A-1}, \\ n \geq k+1. \end{aligned} \tag{3.3.1}$$

Proof. Let $\{x_n, y_n\}$ be arbitrary positive solution of (3.0.1). From (3.0.1) it is obvious that

$$A \leq x_n, \quad B \leq y_n, \quad n \geq 1 \tag{3.3.2}$$

Now, using (3.0.1)and (3.3.2) we get that for all $n \geq 2$

$$\begin{aligned} x_n = A + \frac{x_{n-1}}{y_{n-k-1}} \leq A + \frac{1}{B} x_{n-1}, \\ y_n = B + \frac{y_{n-1}}{x_{n-k-1}} \leq B + \frac{1}{A} y_{n-1}, \\ n \geq k+1. \end{aligned} \tag{3.3.3}$$

Let v_n, w_n be the solution of the system

$$v_{n+1} = A + \frac{1}{B} v_n, \;\; w_{n+1} = B + \frac{1}{A} w_n, \;\; \text{for all } n \geq k+1. \tag{3.3.4}$$

such that

$$v_i = x_i, \quad w_i = y_i, \quad i = -k, -k+1, ..., 0, 1, ..., k+1 \tag{3.3.5}$$

now, we use induction to prove that

$$x_n \leq v_n, \quad y_n \leq w_n, \quad n \geq k+2. \tag{3.3.6}$$

Suppose that (3.3.6) is true for $n = m \geq k+2$. Then from (3.3.3) we get

$$x_{m+1} \leq A + \frac{1}{B}x_m \leq A + \frac{1}{B}v_m = v_{m+1},$$

$$y_{m+1} \leq B + \frac{1}{A}y_m \leq B + \frac{1}{A}w_m = w_{m+1}.$$

Therefore (3.3.6) is true. For simplicity, let $a = \frac{1}{B}, b = A, d = B$ *and* $c = \frac{1}{A}$. Then (3.3.4) becomes

$$v_{n+1} = av_n + b, \quad w_{n+1} = cw_n + d, \quad n \geq k$$

implies that

$$v_{k+l} = a^l v_{k+1} + \frac{1-a^l}{1-a}b, \quad w_{k+l} = c^l w_{k+1} + \frac{1-c^l}{1-c}d$$

since $A > 1, B > 1, a = \frac{1}{B}, b = A, d = B$ and $c = \frac{1}{A}$. Then for $i = k+2, k+3, ..., 3k+3$ and $l \geq 0$ implies

$$\begin{aligned} A \leq x_{k+l} &\leq (\frac{1}{B})^l \left(x_{k+1} - \frac{AB}{B-1} \right) + \frac{AB}{B-1}, \\ B \leq y_{k+l} &\leq (\frac{1}{A})^l \left(y_{k+1} - \frac{AB}{A-1} \right) + \frac{AB}{A-1}, \\ & n \geq k+1. \end{aligned} \tag{3.3.7}$$

The proof is complete. ■

Theorem 3.4. *Suppose that $A > 1, B > 1$. Then the positive equilibrium*

$$(\bar{x}, \bar{y}) = \left(\frac{AB-1}{B-1}, \frac{AB-1}{A-1}\right)$$

of (2.0.1) is locally asymptotically stable.

Proof. System (3.0.1) may be formulated as a system of first order recurrence equations as follows:

$$\begin{aligned} w_n^1 = x_n, w_n^2 = x_{n-1}, ..., w_n^{(k+1)} = x_{n-k} \\ v_n^1 = y_n, v_n^2 = y_{n-1}, ..., v_n^{(k+1)} = y_{n-k} \end{aligned} \tag{3.3.8}$$

Let $Z_n = (w_n^1, w_n^2, ..., w_n^{(k+1)}, v_n^1, v_n^2, ..., v_n^{(k+1)})^T$. Then the linearized system of system (3.0.1) associated with (3.3.8) about the equilibrium point $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ is

$$Z_{n+1} = JZ_n$$

where

$$Z_{n+1} = \begin{pmatrix} w_{n+1}^{(1)} \\ w_{n+1}^{(2)} \\ \vdots \\ w_{n+1}^{(k+1)} \\ v_{n+1}^{(1)} \\ v_{n+1}^{(2)} \\ \vdots \\ v_{n+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} A + \frac{x_n}{y_{n-k}} \\ x_n \\ \vdots \\ x_{n-k+1} \\ x_{n-k} \\ B + \frac{y_n}{x_{n-k}} \\ y_n \\ \vdots \\ y_{n-k+1} \\ y_{n-k} \end{pmatrix}$$

and J is the Jacobian matrix.

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} D_{w_n^{(1)}} Z_{n+1} & \dots & D_{w_n^{(k+1)}} Z_{n+1} & D_{v_n^{(1)}} Z_{n+1} & \dots & D_{v_n^{(k+1)}} Z_{n+1} \end{pmatrix}$$

so the Jacobian matrix will be of the following form

$$J_{(2k+2)\times(2k+2)} = \begin{pmatrix} \frac{1}{\bar{y}} & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{-\bar{x}}{\bar{y}^2} \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \frac{-\bar{y}}{\bar{x}^2} & \frac{1}{\bar{x}} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Let $\lambda_1, \lambda_2, ..., \lambda_{2k+2}$ be the eigenvalues of J. Define $D = diag(d_1, d_2, ..., d_{2k+2})$ be a diagonal matrix such that $d_1 = d_{k+2} = 1, \quad d_{1+k} = d_{m+2+k} = 1 - m\epsilon, \quad 1 \leq m \leq k$ and $\epsilon = \min\left\{\frac{1}{k}, \frac{1}{k}\left(1 - \frac{\bar{x}}{\bar{y}^2 - \bar{y}}\right), \frac{1}{k}\left(1 - \frac{\bar{y}}{\bar{x}^2 - \bar{x}}\right)\right\}$. Clearly, D is invertible. Computing DJD^{-1}, we obtain

$$DJD^{-1} = \begin{pmatrix} \frac{1}{\bar{y}}d_1d_1^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{-\bar{x}}{\bar{y}^2}d_1d_{2k+2}^{-1} \\ d_2d_1^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & & & & & \\ 0 & \cdots & d_{k+1}d_k^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \frac{-\bar{y}}{\bar{x}^2}d_{k+2}d_{k+1}^{-1} & \frac{1}{\bar{x}}d_{k+2}d_{k+2}^{-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & d_{k+3}d_{k+2}^{-1} & \cdots & 0 & 0 \\ & & & & & \ddots & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & d_{2k+2}d_{2k+1}^{-1} & 0 \end{pmatrix}$$

The following two chains of inequalities

$$d_{k+1} > d_k > \cdots > d_2 > 0, \quad d_{2k+2} > d_{2k+1} > \cdots > d_{k+3} > 0$$

imply that

$$d_2d_1^{-1} < 1, d_3d_2^{-1} < 1, \cdots, d_{k+1}d_k^{-1} < 1,$$

$$d_{k+3}d_{k+2}^{-1} < 1, d_{k+4}d_{k+3}^{-1} < 1, \cdots, d_{2k+2}d_{2k+1}^{-1} < 1.$$

Furthermore,

$$\begin{array}{rcl} \frac{d_1}{\bar{y}}d_1^{-1} + \frac{\bar{x}}{\bar{y}^2}d_1 d_{2k+2}^{-1} & = & \frac{1}{\bar{y}} + \frac{\bar{x}}{\bar{y}^2}d_{2k+2}^{-1} \\ & = & \frac{1}{\bar{y}} + \frac{\bar{x}}{\bar{y}^2(1-k\epsilon)} \\ & < & \frac{1}{\bar{y}} + \frac{\bar{x}}{\bar{y}^2(1-k(\frac{1}{k}(1-\frac{\bar{x}}{\bar{y}^2-\bar{y}})))} \\ & = & \frac{1}{\bar{y}} + \frac{\bar{x}}{\bar{y}^2(\frac{\bar{x}}{\bar{y}^2-\bar{y}})} \\ & = & \frac{1}{\bar{y}} + \frac{\bar{y}^2-\bar{y}}{\bar{y}^2} \\ & = & \frac{\bar{y}^2}{\bar{y}^2} \\ & = & 1 \end{array}$$

$$\frac{d_{k+2}}{\bar{x}}d_{k+2}^{-1} + \frac{\bar{y}}{\bar{x}^2}d_{k+2}d_{k+2}^{-1} = \frac{1}{\bar{x}} + \frac{\bar{y}}{\bar{x}^2}d_{k+1}^{-1} = \frac{1}{\bar{x}} + \frac{\bar{y}}{\bar{x}^2(1-k\epsilon)} < 1.$$

It is well known that J has the same eigenvalues as DJD^{-1}, we obtain that

$$\rho(J) = max\{|\ \lambda_i\ |\} \leq \|DJD^{-1}\|_\infty$$

but

$$||DJD^{-1}||_\infty = \left\{ \begin{array}{c} \frac{d_1}{\bar{y}}d_1^{-1} + \frac{\bar{x}}{\bar{y}^2}d_1 d_{2k+2}^{-1}, d_{k+1}d_k^{-1}, d_{k+3}d_{k+2}^{-1}, ..., d_{2k+2}d_{2k+1}^{-1}, d_2 d_1^{-1}, \\ \frac{d_{k+2}}{\bar{x}}d_{k+2}^{-1} + \frac{\bar{y}}{\bar{x}^2}d_{k+2}d_{k+1}^{-1} \end{array} \right\} < 1.$$

So the modulus of every eigenvalue of J is less than one. Hence, the unique equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{AB-1}{B-1}, \frac{AB-1}{A-1}\right)$ of system (3.0.1) is locally asymptotically stable. ■

Theorem 3.5. *[17] If $A > 1$ and $B > 1$, then every positive solution of system (3.0.1) converges to the equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ as $n \to \infty$.*

Proof. From (3.3.1)we have

$$\begin{aligned} L_1 &= \lim_{n\to\infty} \sup x_n, \quad l_1 \lim_{n\to\infty} \inf x_n, \\ L_2 &= \lim_{n\to\infty} \sup y_n, \quad l_2 = \lim_{n\to\infty} \inf y_n, \end{aligned} \tag{3.3.9}$$

where $l_i, L_i \in (0, \infty), i = 1, 2$. Now, system (3.0.1) implies that

$$L_1 \leq A + \frac{L_1}{l_2}, \quad L_2 \leq B + \frac{L_2}{l_1},$$

$$l_1 \geq A + \frac{l_1}{L_2}, \quad l_2 \geq B + \frac{l_2}{L_1}.$$

Which can be written as

$$L_1 l_2 \leq A l_2 + L_1, \quad L_2 l_1 \leq B l_1 + L_2,$$

$$l_1 L_2 \geq A L_2 + l_1, \quad l_2 L_1 \geq B L_1 + l_2.$$

implies

$$B L_1 + l_2 \leq l_2 L_1 \leq A l_2 + L_1, \quad A L_2 + l_1 \leq l_1 L_2 \leq B l_1 + L_2$$

From which we have

$$L_1(B-1) \leq l_2(A-1), \quad L_2(A-1) \leq l_1(B-1) \tag{3.3.10}$$

Since $A > 1$ and $B > 1$ and from (3.3.10) imply that $L_1 L_2 \leq l_1 l_2$ from which it follows that

$$L_1 L_2 = l_1 l_2 \tag{3.3.11}$$

We claim that

$$L_1 = l_1, \quad L_2 = l_2. \tag{3.3.12}$$

Suppose on contrary that $l_1 < L_1$. Then from (3.3.11)we have $L_1 L_2 = l_1 l_2 < L_1 l_2$ and so $L_2 < l_2$, which is a contradiction. So $L_1 = l_1$. Similarly, we can prove that $L_2 = l_2$. Therefore, (3.3.12) are true. From (3.0.1)and (3.3.12) we conclude that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

where (x, y) is the unique positive equilibrium of (3.0.1). This completes the proof of the theorem.

■

Theorem 3.6. *If $A > 1$ and $B > 1$, then the unique positive equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ of system (3.0.1) is globally asymptotically stable.*

Proof. Using theorem (3.4) we conclude that the equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ of system (3.0.1) is locally asymptotically stable, but Theorem (3.5) implies that this equilibrium is a global attractor. Thus, the unique positive equilibrium $(\bar{x}, \bar{y}) = (\frac{AB-1}{B-1}, \frac{AB-1}{A-1})$ of system (3.0.1) is globally asymptotically stable. ■

3.4 Numerical Examples

In this section, we give some numerical examples that represent different cases of dynamical behavior of solutions of (2.0.1) the use of MATLAB to illustrate the results we had in the previous sections.

Example 3.1. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{x_n}{y_{n-5}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-5}}, \quad n = 0, 1, \cdots \tag{3.4.1}$$

with $A = 0.3, B = 0.5$, and the initial conditions $x_{-5} = 0.7, x_{-4} = 9.1, x_{-3} = 0.5, x_{-2} = 9.2, x_{-1} = 0.3, x_0 = 10, y_{-5} = 0.3, y_{-4} = 11.3, y_{-3} = 0.5, y_{-2} = 9.3, y_{-1} = 0.2, y_0 = 11.9$. Then the solution of system (3.4.1) is unbounded since $0 < A < 1$ and $0 < B < 1$ and the initial conditions in Theorem (3.2) are satisfying and the unique positive equilibrium point $(\bar{x}, \bar{y}) = (1.7, 1.21)$ is not globally asymptotically stable (see Figure 1.1, Theorem (3.2)).

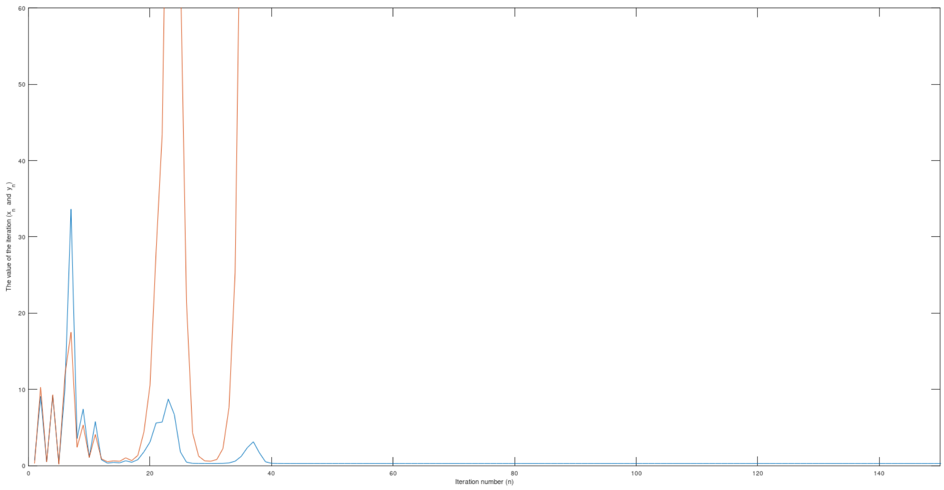

Fig. 3.1: The graph of a solution of system (3.4.1) with $A = 0.3$ and $B = 0.5$

Example 3.2. Consider system (3.4.1) with $A = 4, B = 2.5$, and the initial conditions $x_{-5} = 3.5, x_{-4} = 4.7, x_{-3} = 2.5, x_{-2} = 0.9, x_{-1} = 0.3, x_0 = 0.5, y_{-5} = 3.3, y_{-4} = 4.4, y_{-3} = 2.2, y_{-2} = 0.4, y_{-1} = 0.5, y_0 = 0.7$. Since $A > 1$ and $B > 1$, the solution of system (3.4.1) is bounded and persists (see Theorem (3.3)), and the unique positive equilibrium point $(\bar{x}, \bar{y}) = (6, 3)$ is globally asymptotically stable (see Figure 1.2, Theorem (3.6)).

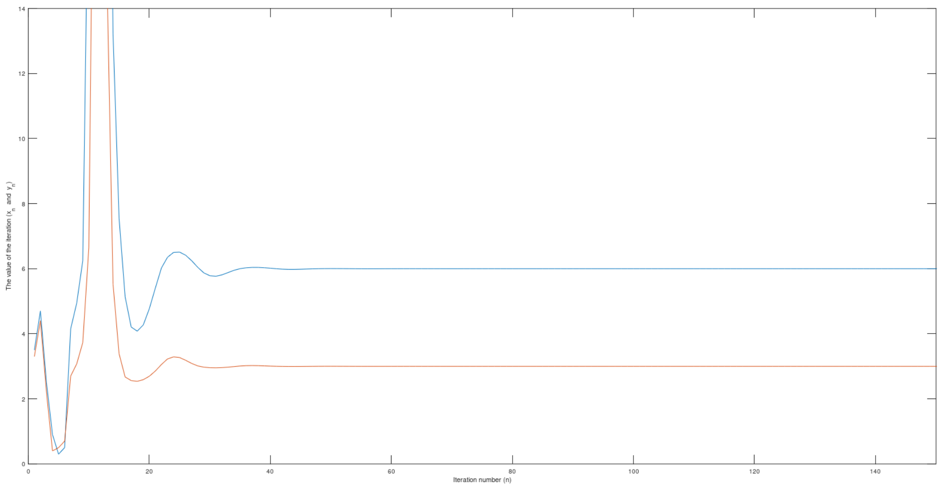

Fig. 3.2: The graph of a solution of system (3.4.1) with $A = 4$ and $B = 2.5$

Example 3.3. Consider the following system of two difference equations:

$$x_{n+1} = A + \frac{x_n}{y_{n-4}}, \quad y_{n+1} = B + \frac{y_n}{x_{n-4}}, \quad n = 0, 1, \cdots \tag{3.4.2}$$

with $A = 3, B = 4$, and the initial conditions $x_{-4} = 0.9, x_{-3} = 2.5, x_{-2} = 2, x_{-1} = 1.1, x_0 = 0.7, y_{-4} = 3.3, y_{-3} = 2, y_{-2} = 0.4, y_{-1} = 0.3, y_0 = 0.9$. Then the unique positive equilibrium point $(\bar{x}, \bar{y}) = (3.7, 5.5)$ is globally asymptotically stable since $A > 1$ and $B > 1$ (see Theorem (3.6)), and the solution of system (3.4.2)is bounded and persists (see Figure 1.3, Theorem (3.3)). In this example $k = 4$ is even, while in Example 1.2, $k = 5$ is odd, but in both cases we have the same conclusion.

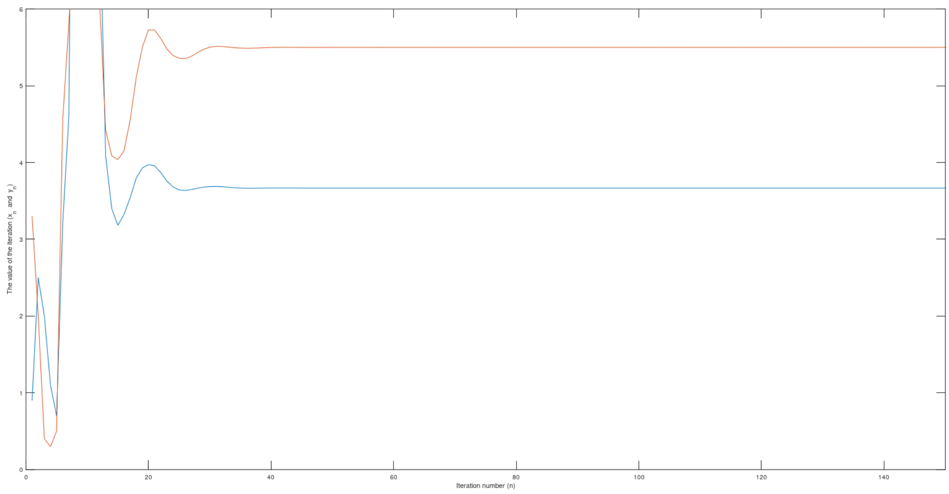

Fig. 3.3: The graph of a solution of system (3.4.2) with $A = 3$ and $B = 4$

CONCLUSION

In this research, we solved an open problem proposed in [1] by Abualrob, S.,Aloqeili, M. We expanded the work on system (1.1.11) to a system with different parameters and investigated its dynamical behavior. We also introduced the symmetrical system of two rational difference equations (3.0.1) and studied the global behavior of its positive solutions.

FUTURE WORK

Our research can be expanded into more complicated related systems. The study of systems (3.0.1), (1.1.1) and (1.1.6) can be extended to systems with distinct parameters. Now, we will give some open problems that can be investigated next.

Problem 1. Investigate the dynamical behavior of the system of two difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-k}}, \quad y_{n+1} = B + \frac{x_n}{y_{n-k}}, \quad n = 0, 1, \cdots \tag{3.4.3}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.

Problem 2. Investigate the dynamical behavior of the system

$$x_{n+1} = A + \frac{y_{n-k}}{x_n}, \quad y_{n+1} = B + \frac{x_{n-k}}{y_n}, \quad n = 0, 1, \cdots \tag{3.4.4}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.

Problem 3. Investigate the dynamical behavior of the system of two nonlinear difference equations

$$x_{n+1} = A + \frac{y^p_{n-k}}{y^q_n}, \quad y_{n+1} = B + \frac{x^p_{n-k}}{x^q_n}, \quad n = 0, 1, \cdots \tag{3.4.5}$$

with parameters $A > 0$ and $B > 0$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.

Problem 4. Investigate the dynamical behavior of the system

$$x_{n+1} = A + \frac{x_{n-k}}{y_n}, \quad y_{n+1} = B + \frac{y_{n-k}}{x_n}, \quad n = 0, 1, \cdots \tag{3.4.6}$$

let $A > 1$ and $B < 1$ or $A < 1$ and $B > 1$, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -k, -k+1, \cdots, 0$ and $k \in Z^+$.

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